The Riemann Zeta Function

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The distribution of prime numbers (2, 3, 5, 7, ... doesn't follow a regular pattern, however, number theorists have been studying the frequency with which prime numbers occur in the set of integers for a long, long time.

While the German mathematician G.F.B. Riemann (1826 - 1866) was investigating the distribution of prime numbers, he observed that the frequency of prime numbers is very closely related to the behavior of a function which has been dubbed the Riemann Zeta (or just zeta) function, and is defined by:

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{u^{z-1}}{e^u - 1}\, du \tag{1}$$

This function has trivial roots at all negative even integers. These roots are trivial because the gamma function $\Gamma(z)$ becomes infinite when z is a negative even integer. However, the zeta function has non-trivial roots as well.

The Riemann hypothesis[1] says that the non-trivial roots have a form of $z = \frac{1}{2} + bi$ for certain b (this is a straight line on the complex plane), and these are the *only* non-trivial roots of this function. Although there have been sketches of what the proof looks like by Riemann and John Nash, there is currently no full and complete proof. People have investigated the hypothesis numerically, and it holds for all numbers up to ridiculous sizes, but this is hardly a proof. The Clay Mathematics Institute is offering a prize of 1 million dollars to the first person who can proove the Riemann hypothesis.

If z is an integer, the zeta function takes on a particularly nice form:

[1] A hypothesis is a mathematical statement that hasn't been proved true.

$$\frac{u^{n-1}}{e^u - 1} = \frac{e^{-u}u^{n-1}}{1 - e^{-u}} = e^{-u}u^{n-1}\sum_{k=0}^{\infty} e^{-ku} = \sum_{k=1}^{\infty} e^{-ku}u^{n-1} \tag{2}$$

So that eq(1) becomes:

$$\zeta(n) = \frac{1}{\Gamma(z)}\sum_{k=1}^{\infty} e^{-ku}u^{n-1}\,du \tag{3}$$

Now the integrand of eq(3) almost looks like the definition of a Gamma function. We can force it to be a Gamma function by letting $y = ku$.

$$\begin{aligned}
\zeta(n) &= \frac{1}{\Gamma(z)}\sum_{k=1}^{\infty}\int_0^{\infty} e^{-y}\left(\frac{y}{k}\right)^{n-1}\frac{dy}{k} \\
&= \frac{1}{\Gamma(z)}\sum_{k=1}^{\infty}\frac{1}{k^n}\underbrace{\int_0^{\infty} e^{-y}y^{n-1}\,dy}_{\Gamma(n)} \\
&= \sum_{k=1}^{\infty}\frac{1}{k^n} \qquad (4)
\end{aligned}$$

This shows that eq(1) is the complex generalization for the Riemann Zeta function, and eq(4) is the special case definition of the Zeta function of a real integer variable. Note that many people take eq(4) as the definition of the zeta function, but this is wrong.

Also note that there is a generalized Riemann Zeta function:

$$\zeta(n, a) = \sum_{k=1}^{\infty}\frac{1}{(n+a)^s} \tag{5}$$

We can use the integral test for divergence to determine some of the points where the zeta function becomes divergent. As discussed, When z is an integer, the zeta function takes on the form:

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$$

Applying the test for divergence,

$$\int x^p \, dx = \begin{cases} \frac{x^{-p+1}}{-p+1} \mid_1^\infty & p \neq 1 \\ \\ \ln(x) \mid_1^\infty & p = 1 \end{cases}$$

This shows a few things. First of all, the zeta function is divergent for all $p \leq 1$ and convergent for all $p > 1$. It also shows that the zeta function reduces to the harmonic series

$$\zeta 1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

which diverges extremely slowly. Astonishingly, the first *million* terms sum to less than 15!