1 Kirchhoff's Diffraction Theory

The basic idea of the Huygens-Fresnel principle is that light disturbance at a point P arises from the superposition of secondary waves that proceed from a surface situated between P and the light source. It was Kirchhoff who first put the Huygens-Fresnel principle on a mathematical basis. The idea is to use Green's Theorem to express components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ at P as a surface integral of the fields and their derivatives about an arbitrary surface surrounding P.

The time dependent wave equation

$$\nabla^{\mathbf{2}} f \;=\; \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

must be satisfied by each component of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$. Here, f represents some component of the electric or magnetic fields.

1.1 Monochromatic Light

First consider the case where f is a component of monochromatic light (one ω):

$$f(x, y, z, t) \;=\; f_o(x, y, z)\, e^{-i\omega t}$$

Since f is a solution to the wave equation, its spatial component, f_o, must satisfy[1] the Helmholtz equation:[2]

$$\left(\nabla^{\mathbf{2}} + k^2\right) f_o \;=\; 0 \qquad \text{where} \quad k = \frac{\omega}{c}$$

Let V be a volume bounded by a closed surface S, and let P be a point within S. Assume that f_o is continuous and has continuous first and second order partials. Let $g(x, y, z, t)$ be *any* other solution to the wave equation with the same boundary conditions. Then the spatial components of f and g, f_o and g_o, must both satisfy Green's Theorem:[3]

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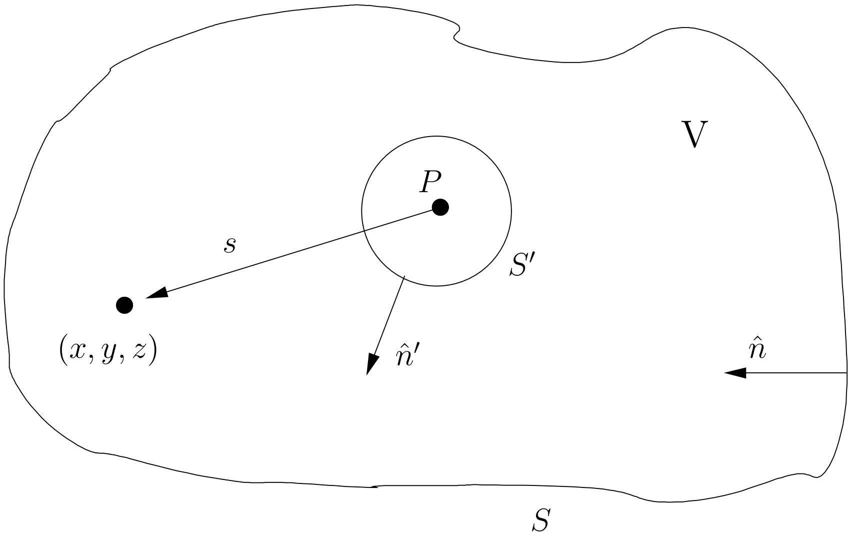
[1] You can show this by plugging f into the wave equation

[2] For this reason, the Helmholtz equation is called the time independent wave equation.

[3] See Jackson section 1.8, but note the change in sign. We're taking the unit normal to S to point *into* the volume V, not outward!



$$\int\int\int_V \left(f_o\nabla^{\mathbf{2}}g_o - g_o\nabla^{\mathbf{2}}f_o\right)\,dV \;=\; \oint_S \left(f_o\frac{\partial g_o}{\partial n} - g_o\frac{\partial f_o}{\partial n}\right)\,dS \tag{1}$$

where $\partial/\partial n$ denotes differentiation inward along a normal to S. Since g is also a solution to the wave equation, g_o must satisfy the Helmholtz equation:

$$\left(\nabla^{\mathbf{2}} + k^2\right)g_o \;=\; 0 \qquad \text{where} \quad k = \frac{\omega}{c}$$

Therefore $f_o\nabla^{\mathbf{2}}g_o - g_o\nabla^{\mathbf{2}}f_o \;=\; f_ok^2g_o - g_ok^2f_o \;=\; 0$ and so the LHS of eqn(1) is zero. Thus, eqn(1) reduces to:

$$\oint_S \left(f_o\frac{\partial g_o}{\partial n} - g_o\frac{\partial f_o}{\partial n}\right)\,dS \;=\; 0 \tag{2}$$

Now take $g_0(x,y,z) = \frac{e^{iks}}{s}$ where s is the distance to P. Because of the singularity at $s = 0$, P must be excluded from the region of integration in the RHS of eqn(1).[4] Therefore, construct a spherical surface S' of radius ϵ around P. The Green Theorem region of integration will be the region between S and S', since this excludes the singularity. Applying Green's Theorem to the volume between S and S':

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[4] Even though the LHS is zero, Green's Theorem only applies to functions with continuous derivatives.

$$\oint_S \left(f_o \frac{\partial g_o}{\partial n} - g_o \frac{\partial f_o}{\partial n} \right) dS - \oint_{S'} \left(f_o \frac{\partial g_o}{\partial n'} - g_o \frac{\partial f_o}{\partial n'} \right) dS' \;=\; 0 \tag{3}$$

Since S' is spherical, it's unit outward normal $\hat{n}'$ is parallel to $\vec{s}$, and therefore $\partial_{\hat{n}'} g_o$ has a simple form:

$$\frac{\partial g_o}{\partial n'} \;=\; \frac{\partial}{\partial n'} \left(\frac{e^{iks}}{s} \right) \;=\; \frac{e^{iks}}{s} \left(ik - \frac{1}{s} \right)$$

and so, from eqn(3) we get:

$$\oint_S \left[f_o \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial f_o}{\partial n} \right] dS \;=\; - \oint_{S'} \left[f_o \frac{e^{iks}}{s} \left(ik - \frac{1}{s} \right) - \frac{e^{iks}}{s} \frac{\partial f_o}{\partial n'} \right] dS' \tag{4}$$

Because S' is a sphere, $dS' = \epsilon^2 \, d\Omega$, and eqn(4) becomes:

$$\oint_S \left[f_o \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial f_o}{\partial n} \right] dS \;=\; - \oint_{\Omega'} \frac{e^{ik\epsilon}}{\epsilon} \left[f_o \left(ik - \frac{1}{\epsilon} \right) - \frac{\partial f_o}{\partial n'} \right] \epsilon^2 \, d\Omega' \tag{5}$$

Clearly, the LHS is independent of ϵ. Therefore, but the RHS must also be independent of ϵ too. Because the whole thing is independent of ϵ, let's take the limit of the RHS as $\epsilon \to 0$.

$$\begin{aligned} &\lim_{\epsilon \to 0} - \oint_{\Omega'} \frac{e^{ik\epsilon}}{\epsilon} \left[f_o \left(ik - \frac{1}{\epsilon} \right) - \frac{\partial f_o}{\partial n'} \right] \epsilon^2 d\Omega' \\ &\qquad = \lim_{\epsilon \to 0} - \oint_{\Omega'} e^{ik\epsilon} \left[f_o \left(ik\epsilon - 1 \right) - \epsilon \frac{\partial f_o}{\partial n'} \right] d\Omega' \\ &\qquad = \oint_{\Omega'} f_o \, d\Omega' \\ &\qquad = 4\pi f_o(P) \end{aligned}$$

That last line comes from the mean value theorem for potentials. See Jackson, problem 1.10, which is a proof of the mean value theorem for potentials. Plugging this result back into eqn(5), we get the Kirchhoff theorem for monochromatic light:

$$f_o(P) \;=\; \frac{1}{4\pi}\oint_S \left[f_o \frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial f_o}{\partial n}\right] dS \tag{6}$$

This theorem expresses a component of $\vec{E}$ or $\vec{H}$ at point P as a surface integral of the field component over any surface surrounding P. One might wonder about this: we need to know the field before we can calculate the field.

In practise, this is not a problem because we often know the field components on bounding surfaces because a physical situation gives us boundary conditions. Therefore, if we know the boundary conditions on field components, this theorem gives us a method of calculating the field at arbitrary points within the boundary.

1.2 Non-Monochromatic Light

Let $f(x, y, z, t)$ be a multichromatic solution to the electromagnetic wave equation (a component of $\vec{E}$ or $\vec{H}$ with many angular frequencies):

$$\nabla^{\mathbf{2}} f \;=\; \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

Since f is multichromatic and we're assuming that it satisfies any continuity conditions that a physical wave must satisfy, it's expressible as a Fourier integral over all angular frequencies:

$$f(x, y, z, t) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_\omega(x, y, z) e^{-i\omega t}\, d\omega \tag{7}$$

Then, by the Fourier inversion formula:

$$F_\omega(x, y, z, w) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x, y, z, t) e^{i\omega t}\, dt$$

By plugging eqn(7) into the wave equation, we can show that if f satisfies the wave equation, F_ω must satisfy the Helmholtz equation. Thus we can apply the Helmholtz theorem to individual Fourier components of F_ω:

$$F_\omega(P) \;=\; F_\omega(x, y, z) \;=\; \frac{1}{4\pi}\oint_S \left[F_\omega \frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial F_\omega}{\partial n}\right] dS \tag{8}$$

Plugging the Helmholtz expression of F_ω eqn(8) into the Fourier expansion for f eqn(7):

$$\begin{aligned}
f(P,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_\omega(x,y,z) e^{-i\omega t}\, d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{4\pi} \oint_S \left[F_\omega \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial F_\omega}{\partial n} \right] \right) e^{-i\omega t}\, dS\, d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \frac{1}{4\pi} \oint_S \left[F_\omega \frac{\partial}{\partial n} \left(\frac{e^{-i\omega(t-s/c)}}{s} \right) - \frac{e^{-i\omega(t-s/c)}}{s} \frac{\partial F_\omega}{\partial n} \right] dS\, d\omega \\
&= \frac{1}{4\pi} \oint_S \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F_\omega \frac{\partial}{\partial n} \left(\frac{e^{-i\omega(t-s/c)}}{s} \right) - \frac{e^{-i\omega(t-s/c)}}{s} \frac{\partial F_\omega}{\partial n} \right] d\omega\, dS \\
&= \frac{1}{4\pi} \oint_S \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F_\omega \left(\frac{\partial}{\partial n} \frac{1}{s} + \frac{i\omega}{sc} \frac{\partial s}{\partial n} \right) \frac{e^{-i\omega(t-s/c)}}{s} - \frac{e^{-i\omega(t-s/c)}}{s} \frac{\partial F_\omega}{\partial n} \right] d\omega\, dS \\
&= \frac{1}{4\pi} \oint_S \frac{1}{\sqrt{2\pi}} \left(\frac{\partial}{\partial n} \frac{1}{s} \right) \int_{-\infty}^{\infty} F_\omega \frac{e^{-i\omega(t-s/c)}}{s}\, d\omega - \frac{1}{sc} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} F_\omega \frac{e^{-i\omega(t-s/c)}}{s}\, d\omega \\
&\qquad - \frac{1}{s} \frac{\partial}{\partial n} \int_{-\infty}^{\infty} F_\omega \frac{e^{-i\omega(t-s/c)}}{s}\, d\omega\, dS
\end{aligned}$$

Where $\int F_\omega e^{-i\omega(t-s/c)}\, d\omega$ is the retarded value[5] of wave equation solution: these are functions that are evaluated at the retarded time $t - s/c$. Therefore, we can write down the general form of Kirchhoff's Theorem:

$$f(P,t) = \frac{1}{4\pi} \oint_S \left[f_{\text{ret}} \frac{\partial}{\partial n} \left(\frac{1}{s} \right) - \frac{1}{sc} \frac{\partial s}{\partial n} \frac{\partial f_{\text{ret}}}{\partial t} - \frac{1}{s} \frac{\partial f_{\text{ret}}}{\partial n} \right] dS \tag{9}$$

Given a solution $f(x,y,z,t)$ (a component of $\vec{E}$ or $\vec{H}$) of the EM wave equation on a closed surface, Kirchoff's Theorem (eqn(6) for monochromatic and eqn(9) for multichromatic light) gives us f at any point within that surface. For multichromatic light, f within S is expressed with a retarded time variable. Strangely, for monochromatic light f at P is not dependent on retarded time. That doesn't make sense to me.

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[5] See any book on electricity and magnetism for an explanation of retarded time.