Verify that

$$\bar{P} = -\int \pi(\bar{x})\, \bar{\nabla} \phi(\bar{x})\, d^3x = \frac{1}{(2\pi)^3} \int_p \vec{p}\, a^\dagger_{\bar{p}} a_{\bar{p}}\, d^3p$$

exactly (without throwing anything away). A small trick in changing variables is required.

$$\int_x \pi(\bar{x}) \vec{\nabla} \phi(\bar{x})\, d^3x$$

$$= \int_x \int_{p'} \frac{d^3p'}{(2\pi)^3} (i) \left[a_{\bar{p}'} - a^\dagger_{-\bar{p}'} \right] \sqrt{\frac{\omega_{\bar{p}'}}{2}}\, e^{i\bar{p}'\cdot\bar{x}}\, \vec{\nabla} \int_p \frac{d^3p}{(2\pi)^3} \left[a_{\bar{p}} + a^\dagger_{-\bar{p}} \right] \frac{1}{\sqrt{2\omega_{\bar{p}}}}\, e^{i\bar{p}\cdot\bar{x}}\, d^3x$$

$$= \frac{(-i)}{2(2\pi)^6} \int_x \int_{\bar{p}'} \int_p \sqrt{\frac{\omega_{\bar{p}'}}{\omega_{\bar{p}}}} \left(a_{\bar{p}'} - a^\dagger_{-\bar{p}'} \right) i\vec{p} \left(a_{\bar{p}} + a^\dagger_{-\bar{p}} \right) \underbrace{e^{i(\bar{p}'+\bar{p})\cdot\bar{x}}}\, d^3p\, d^3p'\, d^3x$$

Upon performing integral over x, this becomes $(2\pi)^3 \delta(\bar{p}'+\bar{p})$.

Use $\delta(\bar{p}'+\bar{p})$ to do $\bar{p}'$ integral: $\vec{p}' \to -\vec{p}$ and $\omega_{\bar{p}'} = \omega_{-\bar{p}} = \omega_{\bar{p}}$ since $E = E(|\bar{p}|)$.

$$= \frac{1}{2(2\pi)^3} \int_p \vec{p} \left(a_{-\bar{p}} a_{\bar{p}} + a_{-\bar{p}} a^\dagger_{-\bar{p}} - a^\dagger_{\bar{p}} a_{\bar{p}} - a^\dagger_{\bar{p}} a^\dagger_{-\bar{p}} \right) d^3p$$

$\updownarrow$ c.o.v.: $\bar{p} \to -\bar{p}$

$$= \frac{1}{2(2\pi)^3} \int_p \vec{p} \left(a_{-\bar{p}} a_{\bar{p}} - a_{\bar{p}} a^\dagger_{\bar{p}} - a^\dagger_{\bar{p}} a_{\bar{p}} - a^\dagger_{\bar{p}} a^\dagger_{-\bar{p}} \right) d^3p$$

Add and subtract $a^\dagger_{\bar{p}} a_{\bar{p}}$.

$$= -\frac{1}{2(2\pi)^3} \int_p \vec{p} \left(2a^\dagger_{\bar{p}} a_{\bar{p}} + a^\dagger_{\bar{p}} a^\dagger_{-\bar{p}} + a_{\bar{p}} a^\dagger_{\bar{p}} - a_{-\bar{p}} a_{\bar{p}} - a^\dagger_{p} a_{p} \right) d^3p$$

get rid of this [ ] by symmetry

$$= -\frac{1}{(2\pi)^3} \int_p \vec{p}\, a^\dagger_{\bar{p}} a_{\bar{p}}\, d^3p - \frac{1}{2(2\pi)^3} \int_p \vec{p} \left(a^\dagger_{\bar{p}} a^\dagger_{-\bar{p}} + a_{\bar{p}} a^\dagger_{\bar{p}} - a_{-\bar{p}} a_{\bar{p}} - a^\dagger_{\bar{p}} a_{\bar{p}} \right) d^3p$$

use symmetry under $p \to -\bar{p}$ to get rid of each term on its own

The Fourier modes, in terms of the scalar field, are given by:

$$a_{\bar{p}} \sim i \int \frac{e^{i\bar{p}\cdot\bar{x}}}{\sqrt{\omega_{\bar{p}}}} \overleftrightarrow{\partial_0} \phi\, d^3x$$

modulo powers of 2π here and there. Thus, we see that:

$$a^\dagger_{-p} = -a_{\bar{p}} \qquad \text{and} \qquad a^\dagger_{\bar{p}} = -a_{-\bar{p}}$$

$\boxed{\text{No!}}$ if this were true both a and $a^\dagger$ would annihilate

when ϕ is a scalar field. Using these relationships to eliminate $a^\dagger_{-p}$ and a_{-p},

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$$\int_{x} \Pi(\bar{x}) \vec{\nabla} \phi(\bar{x})\, d^3x$$

$$= -\frac{1}{(2\pi)^3}\int_{p} \vec{p}\, a_{\bar{p}}^{\dagger} a_{\bar{p}}\, d^3p \;-\; \frac{1}{2(2\pi)^3}\int_{p} \vec{p}\,(\underbrace{-a_{\bar{p}}^{\dagger} a_{\bar{p}} + a_{\bar{p}} a_{\bar{p}}^{\dagger} + a_{\bar{p}}^{\dagger} a_{\bar{p}} - a_{\bar{p}}^{\dagger} a_{\bar{p}}}_{=\,0})\, d^3p$$

$$= -\frac{1}{(2\pi)^3}\int_{p} \vec{p}\, a_{\bar{p}}^{\dagger} a_{\bar{p}}\, d^3p$$

Therefore,

$$\vec{P} = -\int \Pi(\bar{x}) \vec{\nabla} \phi(\bar{x})\, d^3x = \frac{1}{(2\pi)^3}\int_{p} \vec{p}\, a_{\bar{p}}^{\dagger} a_{\bar{p}}\, d^3p$$