Salzman

Problem 1

Starting from

$$\mathcal{H} = \frac{1}{2}\left[\pi^2 + (\bar{\nabla}\phi)^2 + (m\phi)^2\right]$$

show that:

$$H = \int d^3x\, \mathcal{H} = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\bar{p}} \left[a^\dagger_{\bar{p}} a_{\bar{p}} + \frac{1}{2}[a_{\bar{p}}, a^\dagger_{\bar{p}}]\right]$$

where $\omega_{\bar{p}} = E_{\bar{p}} = \sqrt{p^2+m^2}$

$$H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int \pi^2 + (\bar{\nabla}\phi)^2 + (m\phi)^2\, d^3x$$

$$= \frac{1}{2}\int_x \int_{p'} \frac{d^3p'}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\bar{p}'}}{2}}\left[a_{\bar{p}'} - a^\dagger_{-\bar{p}'}\right]e^{i\bar{p}'\cdot\bar{x}} \int_p \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\bar{p}}}{2}}\left[a_{\bar{p}} - a^\dagger_{\bar{p}}\right]e^{i\bar{p}\cdot\bar{x}}\, d^3x$$

$$+ \frac{m^2}{2}\int_x \int_{p'} \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}'}}}\left[a_{\bar{p}'} + a^\dagger_{\bar{p}}\right]e^{i\bar{p}'\cdot\bar{x}} \int_p \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}}\left[a_{\bar{p}} + a^\dagger_{-\bar{p}}\right]e^{i\bar{p}\cdot\bar{x}}\, d^3x$$

$$+ \frac{1}{2}\int_x \int_{p'} \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}'}}}\left[a_{\bar{p}'} + a^\dagger_{-\bar{p}'}\right]\underbrace{\bar{\nabla}e^{i\bar{p}'\cdot\bar{x}}}_{i\bar{p}'} \cdot \int_p \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}}\left[a_{\bar{p}} + a^\dagger_{-\bar{p}}\right]\underbrace{\bar{\nabla}e^{i\bar{p}\cdot\bar{x}}}_{i\bar{p}}\, d^3x$$

$$= -\frac{1}{4(2\pi)^6}\int_x \int_p \int_{p'} \sqrt{\omega_{\bar{p}'}\omega_{\bar{p}}}\left[a_{\bar{p}'} - a^\dagger_{-\bar{p}'}\right]\left[a_{\bar{p}} - a^\dagger_{-\bar{p}}\right]e^{i(\bar{p}+\bar{p}')\cdot\bar{x}}\, d^3p'\, d^3p\, d^3x$$

$$- \frac{1}{4(2\pi)^6}\int_x \int_p \int_{p'} \frac{\vec{p}'\cdot\vec{p}}{\sqrt{\omega_{\bar{p}'}\omega_{\bar{p}}}}\left[a_{\bar{p}'} + a^\dagger_{\bar{p}'}\right]\left[a_{\bar{p}} + a^\dagger_{-\bar{p}}\right]e^{i(\bar{p}+\bar{p}')\cdot x}\, d^3p'\, d^3p\, d^3x$$

$$+ \frac{m^2}{4(2\pi)^6}\int_x \int_p \int_{p'} \frac{1}{\sqrt{\omega_{\bar{p}'}\omega_{\bar{p}}}}\left[a_{\bar{p}'} + a^\dagger_{-\bar{p}'}\right]\left[a_{\bar{p}} + a^\dagger_{-\bar{p}}\right]e^{i(\bar{p}+\bar{p}')\cdot x}\, d^3p'\, d^3p\, d^3x$$

Do integral over x - get $(2\pi)^3\delta(\vec{p}+\vec{p}\,')$. Note that $\omega_{-\vec{p}} = \omega_{\vec{p}}$ since $\omega_{\vec{p}} = E_p = \sqrt{p^2+m^2}$, so $\omega_{\vec{p}}$ should really be $\omega_{|\vec{p}|}$. I'll use the delta function to do the integral over p'.

$$H = -\frac{1}{4(2\pi)^3}\int_p \omega_{\vec{p}}\,[a_{-\vec{p}} - a_{\vec{p}}^\dagger][a_{\vec{p}} - a_{-\vec{p}}^\dagger]\,d^3p$$

$$-\frac{1}{4(2\pi)^3}\int_p \frac{p^2}{\omega_{\vec{p}}}\,[a_{-\vec{p}} + a_{\vec{p}}^\dagger][a_{\vec{p}} + a_{-\vec{p}}^\dagger]\,d^3p$$

should be + since $-p'\cdot p = +p^2$; this is what you used below!

sign

$$+\frac{m^2}{4(2\pi)^3}\int_p \frac{1}{\omega_{\vec{p}}}\,[a_{-\vec{p}} + a_{\vec{p}}^\dagger][a_{\vec{p}} + a_{-\vec{p}}^\dagger]\,d^3p$$

The last two integrals can be combined to give $p^2+m^2 = E_p^2 = \omega_p^2$.

$$H = -\frac{1}{4(2\pi)^3}\int_p \omega_{\vec{p}}\left\{[a_{-\vec{p}} - a_{\vec{p}}^\dagger][a_{\vec{p}} - a_{-\vec{p}}^\dagger] - [a_{-\vec{p}} + a_{\vec{p}}^\dagger][a_{\vec{p}} + a_{-\vec{p}}^\dagger]\right\}d^3p$$

$$= +\frac{1}{4(2\pi)^3}\int_p \omega_{\vec{p}}\left\{\cancel{-a_{-\vec{p}}a_{\vec{p}}} + a_{-\vec{p}}a_{-\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}} \cancel{- a_{\vec{p}}^\dagger a_{-\vec{p}}^\dagger} + \cancel{a_{-\vec{p}}a_{\vec{p}}} + a_{-\vec{p}}a_{-\vec{p}}^\dagger\right.$$

$$\left. + a_{\vec{p}}^\dagger a_{\vec{p}} + \cancel{a_{\vec{p}}^\dagger a_{-\vec{p}}^\dagger}\right\}d^3p$$

COV on the $(-\vec{p})(-\vec{p})$ terms $\vec{p}\to-\vec{p}$. Since d^3p is invariant under this transformation, we can write:

$$H = \frac{1}{4(2\pi)^3}\int_p \omega_{\vec{p}}\left\{2a_{\vec{p}}^\dagger a_{\vec{p}} + 2a_{\vec{p}}a_{\vec{p}}^\dagger\right\}d^3p$$

$$= \frac{1}{2(2\pi)^3}\int_p \omega_{\vec{p}}\left\{a_{\vec{p}}^\dagger a_{\vec{p}} + a_{\vec{p}}a_{\vec{p}}^\dagger - a_{\vec{p}}^\dagger a_{\vec{p}} + a_{\vec{p}}^\dagger a_{\vec{p}}\right\}d^3p$$ add + subtract $a_{\vec{p}}^\dagger a_{\vec{p}}$

$$= \frac{1}{(2\pi)^3}\int_p \omega_{\vec{p}}\left\{a_{\vec{p}}^\dagger a_{\vec{p}} + \tfrac{1}{2}[a_{\vec{p}}, a_{\vec{p}}^\dagger]\right\}d^3p$$ ✓

as required.