Problem 9 10/10

As an explicit check of Wick's Theorem for 4 scalar fields, show explicitly that

$$T[1234] = N[\text{all contractions}]$$

note - I consider "all contractions" to include no contractions.

①

$$T[1234] = 2134 = (\overset{+}{2}+\overset{-}{2})(\overset{+}{1}+\overset{-}{1})(\overset{+}{3}+\overset{-}{3})(\overset{+}{4}+\overset{-}{4})$$

$$= (\overset{+}{2}\overset{+}{1}+\overset{+}{2}\overset{-}{1}+\overset{-}{2}\overset{+}{1}+\overset{-}{2}\overset{-}{1})(\overset{+}{3}\overset{+}{4}+\overset{+}{3}\overset{-}{4}+\overset{-}{3}\overset{+}{4}+\overset{-}{3}\overset{-}{4})$$

$$= \cancel{\overset{+}{2}\overset{+}{1}\overset{+}{3}\overset{+}{4}} + \overset{\text{row 2}}{\cancel{\overset{+}{2}\overset{+}{1}\overset{+}{3}\overset{-}{4}}} + \overset{\text{row 2}}{\cancel{\overset{+}{2}\overset{+}{1}\overset{-}{3}\overset{+}{4}}} + \overset{\text{row 3}}{\cancel{\overset{+}{2}\overset{+}{1}\overset{-}{3}\overset{-}{4}}}$$

$$\overset{\text{row 3}}{\cancel{\overset{+}{2}\overset{-}{1}\overset{+}{3}\overset{+}{4}}} + \overset{\text{row 4}}{\cancel{\overset{+}{2}\overset{-}{1}\overset{+}{3}\overset{-}{4}}} + \cancel{\overset{+}{2}\overset{-}{1}\overset{-}{3}\overset{+}{4}} + \cancel{\overset{+}{2}\overset{-}{1}\overset{-}{3}\overset{-}{4}}$$

$$\cancel{\overset{-}{2}\overset{+}{1}\overset{+}{3}\overset{+}{4}} + \cancel{\overset{-}{2}\overset{+}{1}\overset{+}{3}\overset{-}{4}} + \overset{\text{row 6}}{\cancel{\overset{-}{2}\overset{+}{1}\overset{-}{3}\overset{+}{4}}} + \overset{\text{row 6}}{\cancel{\overset{-}{2}\overset{+}{1}\overset{-}{3}\overset{-}{4}}}$$

$$\cancel{\overset{-}{2}\overset{-}{1}\overset{+}{3}\overset{+}{4}} + \overset{\text{row 7}}{\cancel{\overset{-}{2}\overset{-}{1}\overset{+}{3}\overset{-}{4}}} + \cancel{\overset{-}{2}\overset{-}{1}\overset{-}{3}\overset{+}{4}} + \cancel{\overset{-}{2}\overset{-}{1}\overset{-}{3}\overset{-}{4}}$$

normal ordering: $\overset{+}{n}$ contains $a_p \longrightarrow$ to the right

$\overset{-}{n}$ contains $a_p^+ \longrightarrow$ to the left

$$T[1234] = \cancel{\overset{+}{2}\overset{+}{1}\overset{+}{3}\overset{+}{4}} + \cancel{\overset{-}{2}\overset{-}{1}\overset{-}{3}\overset{-}{4}} + \cancel{\overset{-}{2}\overset{+}{1}\overset{+}{3}\overset{+}{4}} + \cancel{\overset{-}{2}\overset{-}{1}\overset{+}{3}\overset{+}{4}} + \cancel{\overset{-}{2}\overset{-}{1}\overset{-}{3}\overset{+}{4}}$$

$$+ \cancel{\overset{+}{2}\overset{+}{1}\overset{\sqcap}{34}} + \cancel{\overset{+}{2}\overset{+}{1}\overset{-}{4}\overset{+}{3}} + \cancel{\overset{+}{2}\overset{\sqcap}{13}\overset{+}{4}} + \cancel{\overset{+}{2}\overset{-}{3}\overset{+}{1}\overset{+}{4}}$$

$$+ \cancel{\overset{+}{2}\overset{\sqcap}{13}\overset{-}{4}} + \cancel{\overset{+}{2}\overset{-}{3}\overset{+}{1}\overset{-}{4}} + \cancel{\overset{\sqcap}{21}\overset{+}{3}\overset{+}{4}} + \cancel{\overset{-}{1}\overset{+}{2}\overset{+}{3}\overset{+}{4}}$$

$$+ \cancel{\overset{+}{2}\overset{-}{1}\overset{\sqcap}{34}} + \cancel{\overset{+}{2}\overset{-}{1}\overset{-}{4}\overset{+}{3}} + \cancel{\overset{\sqcap}{21}\overset{-}{3}\overset{+}{4}} + \cancel{\overset{-}{1}\overset{+}{2}\overset{-}{3}\overset{+}{4}}$$

$$+ \cancel{\overset{\sqcap}{21}\overset{-}{3}\overset{-}{4}} + \cancel{\overset{-}{1}\overset{+}{2}\overset{-}{3}\overset{-}{4}} + \cancel{\overset{-}{2}\overset{+}{1}\overset{\sqcap}{34}} + \cancel{\overset{-}{2}\overset{+}{1}\overset{-}{4}\overset{+}{3}}$$

$$+ \cancel{\overset{-}{2}\overset{\sqcap}{13}\overset{+}{4}} + \cancel{\overset{-}{2}\overset{-}{3}\overset{+}{1}\overset{+}{4}} + \cancel{\overset{-}{2}\overset{\sqcap}{13}\overset{-}{4}} + \overset{-}{2}\overset{-}{3}\overset{+}{1}\overset{-}{4}$$

$$+ \cancel{\overset{-}{2}\overset{-}{1}\overset{\sqcap}{34}} + \cancel{\overset{-}{2}\overset{-}{1}\overset{-}{4}\overset{+}{3}}$$

Rearrange

$T(1234) = \cancel{2134} + \cancel{2134} + \cancel{2134} + \cancel{2134} + \cancel{2134}$

$+ \cancel{2134} + \cancel{1234} + \cancel{2134} + \cancel{2314}$

$+ \cancel{2134} + \cancel{2143} + 2134 + \cancel{2134}$

$+ 2134 + \cancel{2134} + \cancel{2134} + \cancel{2134}$

$+ \cancel{2143} + 2413 + \cancel{2314} + \cancel{3214}$

$+ \cancel{1324} + \cancel{4132} + 2314 + 3214$

$+ \cancel{2134} + \cancel{1234} + \cancel{2143} + 1243$

$+ \cancel{1234} + \cancel{1324} + \cancel{1234} + 1324$

$+ \cancel{2143} + \cancel{2413} + \cancel{2314} + \cancel{2341}$

0

$T(1234) = \cancel{2134} + \cancel{2134} + \cancel{2134} + \cancel{2134} + \cancel{2134}$

$+ \cancel{3214} + \cancel{1324} + \cancel{2413} + \cancel{2341}$

$+ \cancel{2314} + \cancel{2314} + \cancel{2143} + 2134 + 2134 + 2134$

$+ 2134 + 2134 + 2134 + 2134 + 2134 + 2134$

$+ 2143 + 2314 + 1324 + 4132 + 2134$

$+ 1234 + 2143 + 1234 + 2314 + 1234 + 2143$

$+ 2413 + \cancel{4213} + 2314 + 2341 + 3214 + \cancel{3241}$

$+ 1243 + \cancel{1423} + 1324 + \cancel{1341}$

$3241 + \cancel{3421}$

$$T[1234] =$$

$$\overset{+}{2}\overset{+}{1}\overset{+}{3}\overset{+}{4} + \bar{4}\overset{+}{2}\overset{+}{1}\overset{+}{3} + \bar{3}\overset{+}{2}\overset{+}{1}\overset{+}{4} + \bar{3}\bar{4}\overset{+}{2}\overset{+}{1}$$
$$+ \bar{1}\overset{+}{2}\overset{+}{3}\overset{+}{4} + \bar{1}\bar{4}\overset{+}{2}\overset{+}{3} + \bar{1}\bar{3}\overset{+}{2}\overset{+}{4} + \bar{1}\bar{3}\bar{4}\overset{+}{1}$$
$$+ \bar{2}\overset{+}{1}\overset{+}{3}\overset{+}{4} + \bar{2}\bar{4}\overset{+}{1}\overset{+}{3} + \bar{2}\bar{3}\overset{+}{1}\overset{+}{4} + \bar{2}\bar{3}\bar{4}\overset{+}{1}$$
$$+ \bar{2}\bar{1}\overset{+}{3}\overset{+}{4} + \bar{2}\bar{1}\bar{4}\overset{+}{3} + \bar{2}\bar{1}\bar{3}\overset{+}{4} + \bar{2}\bar{1}\bar{3}\bar{4}$$

compare to the 4x4 matrix at the top of page 1. I even kept it in the same order for you to check easily.

$$+ \cancel{\overset{\sqcap}{21}\overset{+}{3}\overset{+}{4}} + \cancel{\overset{+}{2}\overset{+}{1}\overset{\sqcap}{34}} + \cancel{\bar{2}\bar{1}\overset{\sqcap}{34}} + \cancel{\overset{+}{2}\overset{\sqcap}{13}\overset{+}{4}} + \cancel{\overset{\sqcap}{21}\bar{3}\overset{+}{4}} + \cancel{\overset{\sqcap}{21}\bar{3}\bar{4}}$$
$$+ \cancel{\bar{2}\overset{+}{1}\overset{\sqcap}{34}} + \cancel{\bar{2}\overset{\sqcap}{13}\bar{4}} + \cancel{\bar{2}\overset{\sqcap}{13}\overset{+}{4}} + \cancel{\overset{+}{2}\overset{\sqcap}{14}\overset{+}{3}} + \cancel{\overset{\sqcap}{23}\overset{+}{1}\overset{+}{4}} + \cancel{\overset{\sqcap}{13}\overset{\sqcap}{24}}$$
$$+ \cancel{\bar{4}\overset{\sqcap}{13}\overset{+}{2}} + \cancel{\overset{\sqcap}{21}\overset{\sqcap}{34}} + \cancel{\bar{1}\overset{+}{2}\overset{\sqcap}{34}} + \cancel{\overset{\sqcap}{21}\bar{4}\overset{+}{3}} + \cancel{\bar{1}\overset{\sqcap}{23}\bar{4}} + \cancel{\bar{2}\bar{3}\overset{\sqcap}{14}}$$
$$+ \cancel{\bar{1}\overset{\sqcap}{23}\overset{+}{4}} + \cancel{\bar{2}\overset{\sqcap}{14}\overset{+}{3}} + \cancel{\overset{\sqcap}{24}\overset{+}{1}\overset{+}{3}} + \cancel{\bar{1}\overset{\sqcap}{24}\overset{+}{3}} + \overset{\sqcap}{23}\overset{\sqcap}{14} + \cancel{\bar{1}\bar{3}\overset{\sqcap}{24}}$$
$$+ \cancel{\overset{\sqcap}{23}\bar{4}\overset{+}{1}} + \cancel{\bar{3}\overset{+}{2}\overset{\sqcap}{14}} + \cancel{\bar{3}\overset{\sqcap}{24}\overset{+}{1}}$$

0

$$T[1234] = N[1234]$$
$$+ \overset{\sqcap}{21}\left[\overset{+}{3}\overset{+}{4} + \bar{3}\overset{+}{4} + \bar{3}\bar{4} + \bar{4}\overset{+}{3}\right]$$
$$+ \left[\overset{+}{2}\overset{+}{1} + \bar{2}\bar{1} + \bar{2}\overset{+}{1} + \bar{1}\overset{+}{2}\right]\overset{\sqcap}{34}$$
$$+ \overset{\sqcap}{13}\left[\overset{+}{2}\overset{+}{4} + \bar{2}\bar{4} + \bar{2}\overset{+}{4} + \bar{4}\overset{+}{2}\right]$$
$$+ \left[\overset{+}{2}\overset{+}{3} + \bar{2}\bar{3} + \bar{2}\overset{+}{3} + \bar{3}\overset{+}{2}\right]\overset{\sqcap}{14}$$
$$+ \overset{\sqcap}{23}\left[\overset{+}{1}\overset{+}{4} + \bar{1}\bar{4} + \bar{1}\overset{+}{4} + \bar{4}\overset{+}{1}\right]$$
$$+ \left[\overset{+}{1}\overset{+}{3} + \bar{1}\overset{+}{3} + \bar{1}\bar{3} + \bar{3}\overset{+}{1}\right]\overset{\sqcap}{24}$$
$$+ 1234 + 1234 + 1234$$

$$T[1234] = N[1234]$$

$$+ N\Big[\wick{\c1 1 \c1 2 3 4} + \wick{\c1 1 2 \c1 3 4} + \wick{\c1 1 2 3 \c1 4} + \wick{1 \c1 2 \c1 3 4} + \wick{1 \c1 2 3 \c1 4}$$

$$+ \wick{1 2 \c1 3 \c1 4} + \wick{\c1 1 \c2 2 \c1 3 \c2 4} + \wick{\c1 1 \c1 2 \c2 3 \c2 4} + \wick{\c1 1 \c2 2 \c2 3 \c1 4}\Big]$$

$$= N[\Sigma \text{ all contractions}]$$

Note- of course, I made extensive use of the definition of contraction:

$$\wick{\c1 1 \c1 2} = \begin{cases} [\overset{+}{1}, \overset{-}{2}] & 1^\circ > 2^\circ \\ [\overset{+}{2}, \overset{-}{1}] & 2^\circ > 1^\circ \end{cases}$$