Peskin + Schroeder 3.7 This problem concerns discrete symmetries P, C, T.

a) Compute the transformation properties under P, C, T of the antisymmetric tensor fermion bilinears, $\bar{\psi}\sigma^{\mu\nu}\psi$, with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. This completes the table of the transformation properties of bilinears at the end of the chapter.

b) Let $\phi(x)$ be a complex valued Klein-Gordon field, such as we considered in Problem 2.2. Find unitary operators P, C and an antiunitary operator T (all defined in terms of their action on the annhilation operators a_p, b_p for the KG particles + antiparticles) that give the following transformations of the KG field:

$$P\phi(t,\bar{x})P = \phi(t,-\bar{x}) \qquad T\phi(t,\bar{x})T = \phi(-t,\bar{x})$$

$$C\phi(t,\bar{x})C = \phi^*(t,\bar{x})$$

Find the transformation properties of the components of the current,

$$J^\mu = i(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi)$$

under P, C, T.

c) Show that any hermitian, Lorentz-scalar local operator built from $\psi(x)$, $\phi(x)$, and their conjugates has CPT = +1.

a) Time Reversal

$$T\bar{\Psi}\sigma^{\mu\nu}\psi T = \bar{\Psi}\gamma^1\gamma^3\sigma^{\mu\nu *}(-\gamma^1\gamma^3)\psi = \bar{\Psi}\gamma^1\gamma^3\frac{i}{2}[\gamma^\mu,\gamma^\nu]^*\gamma^1\gamma^3\psi$$

Case: $\mu=0$, $\nu=1$ or 3 (all γ's are real)

$$T\bar{\Psi}\sigma^{\mu\nu}\psi T = \bar{\Psi}\gamma^1\gamma^3\frac{i}{2}(\gamma^0\gamma^{1\text{ or }3} - \gamma^{1\text{ or }3}\gamma^0)\underbrace{\gamma^1\gamma^3}\psi$$

One passes $\sigma^{\mu\nu}$ for free. The other must pay a -1.

$$= -\bar{\Psi}\gamma^1\gamma^3\gamma^1\gamma^3\sigma^{\mu\nu}\psi \quad (-1) \quad = +\bar{\Psi}\underbrace{\gamma^1\gamma^1}_{-1}\underbrace{\gamma^3\gamma^3}_{-1}\sigma^{\mu\nu}\psi$$

$$= \bar{\Psi}\sigma^{\mu\nu}\psi$$

Case: $\mu=0$, $\nu=2$

The γ^1, γ^3 both anticommute with γ^0, γ^2 to give $(-1)(-1) = +1$, but we pick up an extra minus sign in the commutator since $\gamma^{2*} = -\gamma^2$. Thus, we get the same result of $\bar{\Psi}\sigma^{\mu\nu}\psi$.

Case: $\mu=1$, $\nu=2$

$$T\bar{\Psi}\sigma^{\mu\nu}\psi T = \bar{\Psi}\gamma^1\gamma^3\frac{i}{2}\underbrace{(\gamma^1\gamma^2-\gamma^2\gamma^1)^*}_{-1 \text{ after conjugation}}\gamma^1\gamma^3\psi \quad (-1)$$

$$= \bar{\Psi}\gamma^1\gamma^3\gamma^1\frac{i}{2}(\gamma^1\gamma^2-\gamma^2\gamma^1)\gamma^3\psi \quad (-1)^2 = +1$$

$$= \bar{\Psi}\gamma^1\gamma^3\gamma^1\gamma^3\sigma^{\mu\nu}\psi \quad (-1)$$

$$= -\bar{\Psi}\underbrace{\gamma^1\gamma^1}_{-1}\underbrace{\gamma^3\gamma^3}_{-1}\sigma^{\mu\nu}\psi$$

$$= -\bar{\Psi}\sigma^{\mu\nu}\psi$$

Case: $\mu=1,\ \nu=3$ (all γ's are real)

$$T\bar{\Psi}\sigma^{\mu\nu}\psi T = \bar{\Psi}\gamma^1\gamma^3\tfrac{i}{2}[\gamma^1,\gamma^3]^*\underbrace{\gamma^1\gamma^3}_{\text{each must pay a -1 to pass through.}}\psi$$

$$= \bar{\Psi}\gamma^1\overset{-1}{\gamma^3\gamma^1}\gamma^3\sigma^{\mu\nu}\psi$$

$$= -\bar{\Psi}\gamma^1\gamma^1\gamma^3\gamma^3\sigma^{\mu\nu}\psi \qquad (\gamma^1)^2=(\gamma^3)^2=-1$$

$$= -\bar{\Psi}\sigma^{\mu\nu}\psi$$

Case: $\mu=2,\ \nu=3$

$$T\bar{\Psi}\sigma^{\mu\nu}\psi T = \bar{\Psi}\gamma^1\gamma^3\tfrac{i}{2}(\gamma^2\gamma^3-\gamma^3\gamma^2)^*\gamma^1\gamma^3\psi$$

get -1 after conjugation

passes through for free

must pay -1.

$$= \bar{\Psi}\gamma^1\overset{-1}{\gamma^3\gamma^1}\gamma^3\sigma^{\mu\nu}\psi \qquad (\gamma^1)^2=(\gamma^3)^2=-1$$

$$= -\bar{\Psi}\sigma^{\mu\nu}\psi$$

Since $\sigma^{\mu\nu}$ is antisymmetric I've exhausted all possibilities. To summarize, For $\sigma^{\mu\nu}$:

space/space indices : $T\bar{\Psi}\sigma^{\mu\nu}\psi T \to -\bar{\Psi}\sigma^{\mu\nu}\psi$

space/time indices : $T\bar{\Psi}\sigma^{\mu\nu}\psi T \to +\bar{\Psi}\sigma^{\mu\nu}\psi$

Charge Conjugation

First we must know the transpose of the pseudo tensor. Observe that:

$$\gamma^{0T} = \gamma^0 \qquad \gamma^{2T} = \gamma^2 \qquad \gamma^{5T} = \gamma^5$$

$$\gamma^{1T} = -\gamma^1 \qquad \gamma^{3T} = -\gamma^3$$

so:

$$(\sigma^{\mu\nu})^T = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^T = \frac{i}{2}\left[(\gamma^\mu\gamma^\nu)^T - (\gamma^\nu\gamma^\mu)^T\right]$$

$$= \frac{i}{2}\left(\gamma^{\nu T}\gamma^{\mu T} - \gamma^{\mu T}\gamma^{\nu T}\right) \qquad \text{by socks/shoes property}$$

$$= \begin{cases} -\sigma^{\mu\nu} & \mu,\nu \text{ not both } 1,3 \\ -\sigma^{\mu\nu} & \mu,\nu \text{ both } 1,3 \\ +\sigma^{\mu\nu} & \text{otherwise} \end{cases}$$

$$C\bar{\Psi}\sigma^{\mu\nu}\Psi C = (i\gamma^0\gamma^2\Psi)^T\sigma^{\mu\nu}(i\bar{\Psi}\gamma^0\gamma^2)^T = -\Psi^T\gamma^2\gamma^0\sigma^{\mu\nu}\gamma^2\gamma^0\bar{\Psi}^T$$

<u>case</u>: $\mu = 1 \text{ or } 3 \quad \nu = 1 \text{ or } 3$ but $\mu \neq \nu$ (2 cases here)

For this case, γ^2 and γ^0 commute with $\sigma^{\mu\nu}$.

$$C\bar{\Psi}\sigma^{\mu\nu}\Psi C = -\Psi^T\gamma^2\gamma^0\gamma^2\gamma^0\sigma^{\mu\nu}\bar{\Psi}^T$$

$$= +\Psi^T\underbrace{\gamma^2\gamma^2}_{-1}\underbrace{\gamma^0\gamma^0}_{+1}\sigma^{\mu\nu}\bar{\Psi}^T$$

$$= -\Psi^T\sigma^{\mu\nu}\bar{\Psi}^T$$

$$= \bar{\Psi}(\sigma^{\mu\nu})^T\Psi + \cancel{\phi}_{0}$$

$$= -\bar{\Psi}\sigma^{\mu\nu}\Psi$$

<u>case</u>: $\mu = 1 \text{ or } 3 \quad \nu = 0 \text{ or } 2$ (4 cases here)

For this case, one of (γ^2, γ^0) commutes with $\sigma^{\mu\nu}$. The other anticommutes.

$$C\bar{\Psi}\sigma^{\mu\nu}\Psi C = +\Psi^T\gamma^2\gamma^0\gamma^2\gamma^0\sigma^{\mu\nu}\bar{\Psi}^T$$

$$= -\Psi^T\gamma^2\gamma^2\gamma^0\gamma^0\sigma^{\mu\nu}\bar{\Psi}^T$$

$$= +\Psi^T\sigma^{\mu\nu}\bar{\Psi}^T$$

$$= -\bar{\Psi}(\sigma^{\mu\nu})^T\Psi + \cancel{\phi}_{0}$$

$$= -\bar{\Psi}\sigma^{\mu\nu}\Psi$$

By antisymmetry of $\sigma^{\mu\nu}$ I've covered all cases. Thus $C\bar{\Psi}\sigma^{\mu\nu}\Psi C = -\bar{\Psi}\sigma^{\mu\nu}\Psi$.

Parity

$$P\bar{\Psi}\sigma^{\mu\nu}\psi P = \bar{\Psi}\gamma^0\sigma^{\mu\nu}\gamma^0\psi$$

case: μ,ν both spatial (non zero) then γ^0 commutes through $\sigma^{\mu\nu}$.

$$= \bar{\Psi}\gamma^0\gamma^0\sigma^{\mu\nu}\psi$$
$$= \bar{\Psi}\sigma^{\mu\nu}\psi$$

case: $\mu = \begin{pmatrix}\text{spatial} \\ \text{time}\end{pmatrix}$ $\nu = \begin{pmatrix}\text{time} \\ \text{spatial}\end{pmatrix}$ then γ^0 anticommutes through $\sigma^{\mu\nu}$.

$$= -\bar{\Psi}\gamma^0\gamma^0\sigma^{\mu\nu}\psi$$
$$= -\bar{\Psi}\sigma^{\mu\nu}\psi$$

So when both indices on $\sigma^{\mu\nu}$ are spatial,

$$P\bar{\Psi}\sigma^{\mu\nu}\psi P \rightarrow \bar{\Psi}\sigma^{\mu\nu}\psi$$

When one index is spatial and the other is temporal

$$P\bar{\Psi}\sigma^{\mu\nu}\psi P \rightarrow -\bar{\Psi}\sigma^{\mu\nu}\psi$$

b) From what we did with Klein-Gordon:

$$\phi(x) = \phi_1 + i\phi_2 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A_p + B^\dagger_{-p}\right) e^{ip\cdot x} \qquad A = \tfrac{1}{\sqrt{2}}(a_p + i b_p)$$

$$\phi^*(x) = \phi_1^* - i\phi_2^* = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A^\dagger_p + B_{-p}\right) e^{-ip\cdot x} \qquad B = \tfrac{1}{\sqrt{2}}(a^\dagger_p + i b^\dagger_p)$$

For parity, we would naturally expect

$$P A_p P \rightarrow A_{-p}$$

$$P B_p P \rightarrow B_{-p}$$

So:

$$P\phi P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A_{-p} + B^\dagger_p\right) e^{ip\cdot x}$$

change variables $p \rightarrow -p$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A_p + B^\dagger_{-p}\right) e^{-ip\cdot x}$$

?

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A_p + B^\dagger_{-p}\right) e^{ip\cdot x} \cancel{e^{i\pi}}$$

We see that ~~ϕ^*~~ $P\phi(x)P = \cancel{e^{i\pi}}\,\phi(-x)$ which was what I would've guessed to begin with. The exact same calculation for ϕ^*:

$$P\phi^* P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A^\dagger_{-p} + B_p\right) e^{-ip\cdot x}$$

let $p \rightarrow -p$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A^\dagger_p + B_{-p}\right) e^{ip\cdot x}$$

$$= e^{i\pi} \phi^*$$

So:

$$\boxed{\begin{aligned} P\phi P &= e^{i\pi}\phi \\ P\phi^* P &= e^{i\pi}\phi^* \end{aligned}}$$

or is an antiparticle simply negative charge ?
Yes

Charge Conjugation

Charge conjugation takes particle to anti-particle (as opposed to $+ \to -$) so its effect on the Fourier coefficients is expected to be: mmm..

me	Kaku	expression	interpretation
A	a	$\frac{1}{\sqrt{2}}(a_1 + ia_2)$	destroy particle
A^+	a^+	$\frac{1}{\sqrt{2}}(a_1^+ - ia_2^+)$	create particle
B^+	b	$\frac{1}{\sqrt{2}}(a_1 - ia_2)$	destruction operator for antiparticle ?
B	b^+	$\frac{1}{\sqrt{2}}(a_1^+ + ia_2^+)$	creation operator for antiparticle ?

your B^+ creates antiparticle

So we'd expect (in Kaku's notation) $Ca^+C = b^+$ and $CaC = b$. In my notation:

$$CA^+C = B \qquad CB^+C = A$$
$$CAC = B^+ \qquad CBC = A^+$$

C should not change $A^+ \to B$ rather $A^+ \to B^+$

so:

$$C\phi C = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left[B_p^+ + A_{-p} \right] e^{ip\cdot x}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left[B_{-p}^+ + A_p \right] e^{-ip\cdot x}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left[A_p + B_{-p}^+ \right] e^{ip\cdot x} e^{i\pi}$$

I'm confident this is wrong. Either my interpretations were wrong or my assumed action of C was wrong.

If for some reason we were to expect

$$CA^\dagger C = B^\dagger \qquad CBC = B^\dagger$$
$$CAC = B \qquad CB^\dagger C = B$$

these are not self-consistent with

then

$$C\phi C = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(B_p + A^\dagger_{-p} \right) e^{ip\cdot x}$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A^\dagger_p + B_{-p} \right) e^{-ip\cdot x}$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A_p + B^\dagger_{-p} \right)^* \left(e^{ip\cdot x} \right)^*$$
$$= \phi^*$$

this is what you want

This is what I expect!

If this is correct, $C\phi^* C = \phi$ follows almost immediately.

Time Reversal

Just as with the vector field, we require

① $T a_p T = a_{-p}$ and $T b_p T = b_{-p}$

② $T z = z^* T$

But unlike the vector field case, we have no spin to reverse. So this should be substantially more easy.

$$\underset{(t,\vec{x})}{T \phi T} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A_{-p} + B_{+p}^{\dagger} \right) e^{-ip\cdot x}$$

$p \to -p$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A_p + B_{-p}^{\dagger} \right) e^{ip\cdot x}$$

$$= \phi\,(-t, \vec{x})$$

You need to get time dependence into the ϕ

$\phi = \int \dots \left(A_p e^{-ip\cdot x} + B_p^{\dagger} e^{ip\cdot x} \right)$

$p\cdot x = Et - \vec{p}\cdot\vec{x}$

variable change only changes $\vec{p}\cdot\vec{x}$ part

Similarly, $T \phi^* T = \phi^*$.

I don't know if this is correct, but it might be plausible. The Klein Gordon equation is 2nd order in time, ~~so~~ whereas Dirac and Schrödinger are 1st order in time derivative.