Peskin + Schroeder 3.6 Fierz Transformations

Let u_i $(i=1,\cdots,4)$ be four 4-component Dirac spinors. In the text we proved the Fierz rearrangement formulae (3.78) and (3.79). The 1st of these formulae can be written in 4-component notation as:

$$\bar{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_2\bar{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right) = -\bar{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_4\bar{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_2$$

In fact, there are similar rearrangement formulae for any product

$$(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4)$$

where Γ^A, Γ^B are any of the 16 combinations of Dirac matrices listed in section 3.4.

a) To begin, normalize the 16 matrices Γ^A to the convention

$$\mathrm{Tr}[\Gamma^A\Gamma^B] = 4\delta^{AB}$$

This gives $\Gamma^A = \{1, \gamma^0, i\gamma^j, \cdots\}$; write all 16 elements of this set.

b) Write the general Fierz identity as an equation

$$(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \sum_{C,D} C^{AB}{}_{CD}(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2)$$

with unknown coefficients $C^{AB}{}_{CD}$. Using the completeness of the 16 Γ^A matrices, show that

$$C^{AB}{}_{CD} = \frac{1}{16}\mathrm{Tr}[\Gamma^C\Gamma^A\Gamma^D\Gamma^B]$$

c) Work out explicitly the Fierz transformation laws for the products $(\bar{u}_1u_2)(\bar{u}_3u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$.

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a) We have 5 bilinears:

1. scalar $\Gamma^S = \mathbb{1}_{4\times4}$
2. pseudoscalar $\Gamma^{PS} = \gamma^5$
3. vector $\Gamma^V = \gamma^\mu$
4. pseudovector $\Gamma^{PV} = \gamma^\mu\gamma^5$
5. tensor $\Gamma^T = \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

scalar

$$Tr(\Gamma^S\Gamma^S) = Tr(\mathbb{1}_{4\times4}) = 4$$ already normalized

pseudoscalar

$$Tr(\Gamma^{PS}\Gamma^{PS}) = Tr[\gamma^5\gamma^5] = Tr[\mathbb{1}_{4\times4}] = 4$$ already normalized.

vector

$$Tr(\Gamma^V\Gamma^V) = \begin{cases} Tr[\gamma^0\gamma^0] = Tr[\mathbb{1}_{4\times4}] = 4 \\ Tr[\gamma^i\gamma^i] = Tr[-\mathbb{1}_{4\times4}] = -4 \end{cases}$$

pseudovector

$$Tr(\gamma^\mu\gamma^5\gamma^\mu\gamma^5) = -Tr(\gamma^\mu\gamma^5\gamma^5\gamma^\mu) = -Tr(\gamma^\mu\gamma^\mu)$$

$$= \begin{cases} 4 & \mu = 1,2,3 \\ -4 & \mu = 0 \end{cases}$$

tensor

$$Tr(\Gamma^T\Gamma^T) = Tr(\sigma^{\mu\nu}\sigma^{\mu\nu}) = -\frac{1}{4}Tr[(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)]$$

$$= Tr[\gamma^\mu\gamma^\nu\gamma^\nu\gamma^\mu]$$ note: we already know $\mu \neq \nu$.

$$= \begin{cases} 4 & \mu,\nu \text{ both spatial} \\ -4 & \text{otherwise (but not equal)} \end{cases}$$

So here's how I propose to redefine the bilinears.

<u>scalar</u> no change

<u>pseudoscalar</u> no change

<u>vector</u>

$$\Gamma^V = \begin{cases} \gamma^0 & \mu = 0 \\ i\gamma^\mu & \mu = 1,2,3 \end{cases}$$

<u>pseudovector</u>

$$\Gamma^{PV} = \begin{cases} i\gamma^0\gamma^5 & \mu = 0 \\ \gamma^\mu\gamma^5 & \mu = 1,2,3 \end{cases}$$

<u>tensor</u>

$$\Gamma^T = \begin{cases} \frac{i}{2}[\gamma^\mu, i\gamma^\nu] & \text{if } \mu = \text{spatial},\ \nu = 0 \\ \frac{i}{2}[\gamma^\mu, \gamma^\nu] & \text{if } \mu = \text{spatial},\ \nu = \text{spatial} \\ \frac{i}{2}[i\gamma^\mu, \gamma^\nu] & \text{if } \mu = 0,\ \nu = \text{spatial} \end{cases}$$

Alternatively, we could do something like:

Define $$\underset{\sim}{\gamma}^\mu = \begin{cases} \gamma^0 & \mu = 0 \\ i\gamma^\mu & \mu = 1,2,3 \\ \gamma^5 & \mu = 5 \end{cases}$$

Redefine $\Gamma^{PV} \equiv i\underset{\sim}{\gamma}^\mu\gamma^5$

Then our normalized bilinears are:

scalar: $\Gamma^S = \mathbb{1}$

pseudoscalar: $\Gamma^{PS} = \underset{\sim}{\gamma}^5$

vector: $\Gamma^V = \underset{\sim}{\gamma}^\mu$

pseudovector: $\Gamma^{PV} = i\underset{\sim}{\gamma}^\mu\underset{\sim}{\gamma}^5$

tensor: $\Gamma^T = \sigma^{\mu\nu} = \frac{i}{2}[\underset{\sim}{\gamma}^\mu, \gamma^\nu]$

Thus, our 16 normalized bilinears are:

$$
\begin{array}{cccc}
\mathbb{1} & \gamma^5 & \gamma^0 & i\gamma^1 \\
i\gamma^2 & i\gamma^3 & i\gamma^0\gamma^5 & \gamma^1\gamma^5 \\
\gamma^2\gamma^5 & \gamma^3\gamma^5 & \frac{i}{2}[\gamma^1, \gamma^2] & \frac{i}{2}[\gamma^1, \gamma^3] \\
\frac{i}{2}[\gamma^2, \gamma^3] & \frac{i}{2}[i\gamma^0, \gamma^1] & \frac{i}{2}[i\gamma^0, \gamma^2] & \frac{i}{2}[i\gamma_0, \gamma^3]
\end{array}
$$

S^{AB} ie $A \neq B$ proof?

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b) Some facts about Dirac matrices:

(i) $Tr[\Gamma^A] = 0 \quad (A \neq 1)$

(ii) $\Gamma^A \Gamma_A = \mathbb{1}$

(iii) $Tr(\Gamma_A \Gamma^B) = 4\delta_A^{\ B}$ ← we imposed this normalization

The 3rd fact shows that the bilinears are complete and form a basis for all 4×4 matrices. In particular, for some arbitrary 4×4 matrix M,

$$M = \sum_{A=1}^{16} C_A \Gamma^A$$

Multiplying through by Γ_B and taking the trace of both sides,

$$Tr[M\Gamma_B] = \sum_{A=1}^{16} C_A Tr[\Gamma^A \Gamma_B] = 4C_B$$

where I made use of property (iii). Therefore, for <u>any</u> arbitrary matrix M we can write:

$$M = \sum_{A=1}^{16} C_A \Gamma^A = \frac{1}{4}\sum_{A=1}^{16} Tr(M\Gamma_A)\Gamma^A$$

In component notation: $[A][B] = A_{ij}B_{jk} = C_{ik}$, so the diagonal elements of C is given by $C_{ii} = A_{ij}B_{ji}$ (sum on J). Writing this last expression in component notation,

$$M_{ik} = \frac{1}{4}\sum_A \underbrace{M_{\ell m}\Gamma_{A m\ell}}_{Trace} \Gamma^A_{ik}$$

Now suppose we had <u>two</u> such arbitrary matrices, say, M times N?

We could write such a product as:

$$MN = \sum_{C=1}^{16} C_C \Gamma^C \sum_{D=1}^{16} C_D \Gamma^D$$

$$= \frac{1}{16} \sum_{C,D=1}^{16} Tr(M\Gamma_C)\, Tr(N\Gamma_D)\, \Gamma^C \Gamma^D$$

Or, in component form,

$$M_{ik} N_{pq} = \frac{1}{16} \sum_{C,D=1}^{16} M_{\ell m} \Gamma_{C\, m\ell}\; N_{rs} \Gamma_{D\, sr}\;\; \Gamma^C_{ik} \Gamma^D_{pq}$$

Now let $M = \Gamma^A$ and $N = \Gamma^B$.

$$\Gamma^A_{ik} \Gamma^B_{pq} = \frac{1}{16} \sum_{CD} \Gamma^A_{\ell m} \Gamma^C_{m\ell}\; \Gamma^B_{rs} \Gamma^D_{sr}\;\; \Gamma^C_{ik} \Gamma^D_{pq}$$

Multiply through by $\bar{U}_1^i U_2^k \bar{U}_3^p U_4^q$

$$(\bar{U}_1^i \Gamma^A_{ik} \bar{U}_2^k)(\bar{U}_3^p \Gamma^B_{pq} U_4^q)$$

$$= \frac{1}{16} \sum_{CD} Tr(\Gamma^A \Gamma^C)\, Tr(\Gamma^B \Gamma^D)$$

Well, you on the right track kind of, but you didn't quite make it.

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c) For the 1st one, $\Gamma^A = \Gamma^B = \mathbb{1}$, so $C^{AB}_{CD} = \frac{1}{16} Tr(\Gamma^C \Gamma^D)$. Since we're using the newly normalized bilinears $C^{AB}_{CD} = \frac{1}{4}\delta_{CD}$. This greatly reduces our work.

$$(\bar{U}_1 U_2)(\bar{U}_3 U_4) = \frac{1}{4}\{\bar{U}_1 U_4 \bar{U}_3 U_2 + \bar{U}_1 \gamma^5 U_4 \bar{U}_3 \gamma^5 U_2 + \bar{U}_1 \gamma^0 U_4 \bar{U}_3 \gamma^0 U_2 - \bar{U}_1 \gamma^1 U_4 \bar{U}_3 \gamma^1 U_2 - \bar{U}_1 \gamma^2 U_4 \bar{U}_3 \gamma^2 U_2$$
$$- \bar{U}_1 \gamma^3 U_4 \bar{U}_3 \gamma^3 U_2 - \bar{U}_1 \gamma^0\gamma^5 U_4 \bar{U}_3 \gamma^0\gamma^5 U_2$$
$$+ \bar{U}_1 \gamma^1\gamma^5 U_4 \bar{U}_3 \gamma^1\gamma^5 U_2 + \bar{U}_1 \gamma^2\gamma^5 U_4 \bar{U}_3 \gamma^2\gamma^5 U_2$$
$$+ \bar{U}_1 \gamma^3\gamma^5 U_4 \bar{U}_3 \gamma^3\gamma^5 U_2 - \bar{U}_1 \gamma^1\gamma^2 U_4 \bar{U}_3 \gamma^1\gamma^2 U_2$$
$$- \bar{U}_1 \gamma^1\gamma^3 U_4 \bar{U}_3 \gamma^1\gamma^3 U_2 - \bar{U}_1 \gamma^2\gamma^3 U_4 \bar{U}_3 \gamma^2\gamma^3 U_2$$
$$+ \bar{U}_1 \gamma^0\gamma^1 U_4 \bar{U}_3 \gamma^0\gamma^1 U_2 + \bar{U}_1 \gamma^0\gamma^2 U_4 \bar{U}_3 \gamma^0\gamma^2 U_2$$
$$+ \bar{U}_1 \gamma^0\gamma^3 U_4 \bar{U}_3 \gamma^0\gamma^3 U_2 .\}$$

For the 2nd one $\Gamma^A = \gamma^\mu$ and $\Gamma^B = \gamma_\mu$. I suppose $\gamma^0 = \gamma_0$ and $\gamma^i = -\gamma_i$ like normal 4-vector components. The coefficient is:

$$C^{AB}_{CD} = \frac{1}{16} Tr[\Gamma^C \Gamma^A \Gamma^D \Gamma^B] = \frac{1}{16} Tr[\Gamma^C \gamma^\mu \Gamma^D \gamma_\mu]$$
$$C^{AB}_{11} = \frac{1}{16} Tr[\gamma^\mu \gamma_\mu] = \frac{1}{16} Tr[4\mathbb{1}] = \frac{1}{4}$$
$$C^{AB}_{12} = \frac{1}{16} tr[\mathbb{1}\gamma^\mu \gamma^5 \gamma_\mu] = -\frac{1}{16} Tr[\gamma^\mu \gamma_\mu \gamma^5] = 0$$
$$C^{AB}_{13} = \frac{1}{16}$$