Problem 2.2 The complex scalar field

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi)\, d^4x$$

It is easiest to analyze this theory by considering $\phi(\bar{x})$ and $\phi^*(\bar{x})$, rather than the real and imaginary parts of $\phi(x)$ as the basic dynamical variables.

a) Find the conjugate momenta to $\phi(\bar{x})$ and $\phi^*(\bar{x})$ and the canonical commutation relations. Show that the hamiltonian is:

$$H = \int d^3x\, (\pi^* \pi + \bar{\nabla}\phi^* \cdot \bar{\nabla}\phi + m^2 \phi^* \phi)$$

Compute the Heisenberg equation of motion for $\phi(\bar{x})$ and show that it is indeed the Klein-Gordon equation.

b) Diagonalize H by introducing creation + annhilation operators. Show that the theory contains 2 sets of particles of mass m.

c) Rewrite the conserved charge

$$Q = \int d^3x\, \frac{i}{2}(\phi^* \pi^* - \pi \phi)$$

in terms of creation/annhilation operators and evaluate the charge of the particles of each type.

$S = \int L\, dt = \int \mathcal{L}\, d^4x$, so the Lagrangian density is:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$
$$= \dot{\phi}^* \dot{\phi} - \underbrace{\bar{\nabla}\phi^* \cdot \bar{\nabla}\phi}_{|\bar{\nabla}\phi|^2} - m^2 \phi^* \phi$$

a) The canonical momentum which is conjugate to ϕ^* and ϕ is:

$$\boxed{\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*} \quad \text{and} \quad \boxed{\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}}$$

In direct analogy with the real scalar field, the canonical commutation relations are given by: (and this is by fiat)

$$[\phi(\bar{x}), \pi(\bar{y})] = [\phi^*(\bar{x}), \pi^*(\bar{y})] = i\, \delta^3(\bar{x}-\bar{y})$$

$$[\phi(\bar{x}), \phi^*(\bar{y})] = [\pi(\bar{x}), \pi^*(\bar{y})] = [\phi(\bar{x}), \pi^*(\bar{y})] = [\phi^*(\bar{x}), \pi(\bar{y})] = 0$$

The hamiltonian is given by the continuum limit of the standard Legendre transformation.

$$H = \int (\pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L})\, d^3x$$

$$= \int (\pi \pi^* + \pi^* \pi - \underbrace{\dot{\phi}^* \dot{\phi}}_{\pi \pi^*} + \bar{\nabla}\phi^* \cdot \bar{\nabla}\phi + m^2 \phi^* \phi)\, d^3x$$

$$= \int (\pi^* \pi + \bar{\nabla}\phi^* \cdot \bar{\nabla}\phi + m^2 \phi^* \phi)\, d^3x$$

as required.

From the Heisenberg equations of motion:

$$i\frac{\partial\phi}{\partial t} = [\phi(\bar{x},t), H] = [\phi(\bar{x},t), \int\{\pi^*(\bar{x}',t)\pi(\bar{x}',t) + \underbrace{\text{function of } \phi}_{\text{zero}}\}d^3x']$$

$$= \int\pi^*(\bar{x}',t)\,[\phi(\bar{x},t), \pi(\bar{x}',t)]\,d^3x'$$

$$= i\pi^*(\bar{x},t)$$

For the next one I use integration by parts and assume ϕ, $\bar{\nabla}\phi \to 0$ as $|\bar{x}| \to \infty$.

$$i\frac{\partial\pi}{\partial t} = [\pi(\bar{x},t), H] = [\pi(\bar{x},t), \int\{\underbrace{\text{function of } \pi}_{\text{zero}} + \bar{\nabla}\phi^*\cdot\bar{\nabla}\phi + m^2\phi^*\phi\}d^3x']$$

$$= [\pi(\bar{x},t), \int\phi(\bar{x}',t)(-\bar{\nabla}^2+m^2)\phi^*d^3x' + \text{surface term}]$$

I spoke too quickly. The surface term is simply a number and π commutes with numbers.

$$i\frac{\partial\pi}{\partial t} = \int[\pi(\bar{x},t), \phi(\bar{x}',t)](-\bar{\nabla}^2+m^2)\phi^*(\bar{x}',t)\,d^3x'$$

$$= -i(-\bar{\nabla}^2+m^2)\phi^*(\bar{x},t)$$

Similar calculations will yield very similar expressions:

$$\dot{\phi}^* = \pi \qquad \text{and} \qquad \dot{\pi}^* = (\bar{\nabla}^2-m^2)\phi$$

Combining these results, we see that ϕ and ϕ^* must each obey the Klein-Gordon equation separately...

$$\dot{\pi} = (\bar{\nabla}^2-m^2)\phi^*$$

$$\frac{\partial^2\phi^*}{\partial t^2} = (\bar{\nabla}^2-m^2)\phi^*$$

$$(\partial_\mu\partial^\mu + m^2)\phi^* = 0$$

And similarly for ϕ.

b) Let $\phi(\bar{x}) = \phi_1(x) + i\phi_2(\bar{x})$ where:

$$\phi_1(\bar{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p + a^\dagger_{-p}\right) e^{i\bar{p}\cdot\bar{x}}$$

$$\phi_2(\bar{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p + b^\dagger_{-p}\right) e^{i\bar{p}\cdot\bar{x}}$$

We already figured out commutations since ϕ_1 and ϕ_2 are both real:

$$[a_{\bar{p}}, a^\dagger_{\bar{p}'}] = [b_{\bar{p}}, b^\dagger_{\bar{p}'}] = (2\pi)^3 \delta(\bar{p} - \bar{p}')$$

all other commutations vanish. So for ϕ and ϕ^* we have:

$$\phi(\bar{x}) = \frac{\phi_1 + i\phi_2}{\sqrt{2}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\frac{a_{\bar{p}} + ib_{\bar{p}}}{\sqrt{2}} + \frac{a^\dagger_{-\bar{p}} + ib^\dagger_{-\bar{p}}}{\sqrt{2}}\right) e^{i\bar{p}\cdot\bar{x}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(A_p + B^\dagger_{-\bar{p}}\right) e^{i\bar{p}\cdot\bar{x}}$$

fix correct normalization

$$\phi^*(\bar{x}) = \frac{\phi_1^* - i\phi_2^*}{\sqrt{2}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\frac{a^\dagger_{\bar{p}} - ib^\dagger_{\bar{p}}}{\sqrt{2}} + \frac{a_{-\bar{p}} - ib_{-p}}{\sqrt{2}}\right) e^{-i\bar{p}\cdot\bar{x}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(A^\dagger_{\bar{p}} + B_{-\bar{p}}\right) e^{-i\bar{p}\cdot\bar{x}}$$

where $A_{\bar{p}} = a_{\bar{p}} + ib_{\bar{p}}$ and $B^\dagger_{-\bar{p}} = a^\dagger_{-\bar{p}} + ib^\dagger_{-\bar{p}}$. What about commutation between A and B?

$$[A_{\bar{p}}, B^\dagger_{\bar{p}'}] = [a_{\bar{p}} + ib_{\bar{p}}, a^\dagger_{\bar{p}'} + i b^\dagger_{\bar{p}'}]$$

$$= [a_{\bar{p}}, a^\dagger_{\bar{p}'}] + i\underbrace{\cancel{[b_{\bar{p}}, a^\dagger_{\bar{p}'}]}}_{0} + i\underbrace{\cancel{[a_{\bar{p}}, b^\dagger_{\bar{p}'}]}}_{0} - [b_{\bar{p}}, b^\dagger_{\bar{p}'}]$$

$$= 0$$

$$[A_{\bar p}, B_{\bar p'}] = [a_{\bar p} + ib_{\bar p},\ a_{\bar p'} - ib_{\bar p'}]$$
$$= [a_{\bar p}, a_{\bar p'}] + i[b_{\bar p}, a_{\bar p'}] + i[a_{\bar p}, b_{\bar p'}] + [b_{\bar p}, b_{\bar p'}]$$
$$= 0 \quad + \quad 0 \quad + \quad 0 \quad + \quad 0$$
$$= 0$$

It's obvious that $[A, A] = [B, B] = 0$.

$$[A_{\bar p}, A^{\dagger}_{\bar p'}] = \left[\frac{a_{\bar p} + ib_{\bar p}}{\sqrt{2}},\ \frac{a^{\dagger}_{\bar p} - ib^{\dagger}_{\bar p}}{\sqrt{2}}\right]$$
$$= [a_{\bar p}, a^{\dagger}_{\bar p}] - i[a_{\bar p}, b^{\dagger}_{\bar p}] + i[b_{\bar p}, a^{\dagger}_{\bar p}] + [b_{\bar p}, b^{\dagger}_{\bar p}] \qquad + \frac{1}{2}$$
$$= \cancel{2}(2\pi)^3\, \delta(\bar p - \bar p')$$

This seems to suggest that

OK

$$\boxed{\begin{aligned} A_p &= \frac{1}{\sqrt{2}}(a_{\bar p} + ib_{\bar p}) & A^{\dagger}_p &= \frac{1}{\sqrt{2}}(a^{\dagger}_{\bar p} - ib^{\dagger}_{\bar p}) \\ B^{\dagger}_p &= \frac{1}{\sqrt{2}}(a^{\dagger}_p + ib^{\dagger}_{\bar p}) & B_p &= \frac{1}{\sqrt{2}}(a_{\bar p} - ib_{\bar p}) \end{aligned}}$$

$$\boxed{[A_{\bar p}, A^{\dagger}_{\bar p'}] = [B_{\bar p}, B^{\dagger}_{\bar p'}] = (2\pi)^3 \delta^3(\bar p - \bar p')}$$

all others vanish

$$\boxed{[a_{\bar p}, a^{\dagger}_{\bar p'}] = [b_{\bar p}, b^{\dagger}_{\bar p'}] = (2\pi)^3 \delta^3(\bar p - \bar p')}$$

all others vanish.

$$\boxed{\begin{aligned} \phi(\bar x) &= \phi_1 + i\phi_2 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A_{\bar p} + B^{\dagger}_{-\bar p}\right) e^{i\bar p \cdot \bar x} \\ \phi^*(\bar x) &= \phi_1^* - i\phi_2^* = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} \left(A^{\dagger}_{\bar p} + B_{-\bar p}\right) e^{-i\bar p \cdot \bar x} \end{aligned}}$$

It's clear that $[\phi(\bar{x}), \phi(\bar{y})] = 0$. What about:

$$[\phi(\bar{x}), \phi^*(\bar{y})] = \int \frac{d^3p}{(2\pi)^6} \frac{1}{\sqrt{\omega_p \omega_{p'}}} \Big(A_{\bar{p}} A^+_{\bar{p}} + \cancel{A_{\bar{p}} B_{-\bar{p}}} + \cancel{B^+_{-\bar{p}} A^+_{\bar{p}}} + B^+_{-\bar{p}} B_{-\bar{p}}$$

$$- A^+_{\bar{p}} A_{\bar{p}} - \cancel{B_{-\bar{p}} A_{\bar{p}}} - \cancel{B^+_{-\bar{p}} A^+_{\bar{p}}} - B_{-\bar{p}} B^+_{-\bar{p}} \Big) e^{i(\bar{p}\cdot\bar{x}) - i\bar{p}\cdot\bar{y}}$$

The remaining 4 terms sum to zero. What about Π? Analogous to what we did in class for the uncharged scalar field,

$$\Pi(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\omega_p}\, (-i) \left(A_{\bar{p}} - B^+_{-\bar{p}} \right) e^{i\bar{p}\cdot\bar{x}}$$

$$\Pi^*(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\omega_p}\, (i) \left(A^+_{p} - B_{-p} \right) e^{-i\bar{p}\cdot\bar{x}}$$

not quite since $\pi(x) = \dot{\phi}^*$ you have these reversed

again, this comes straight from my definition of A, B. From the above calculation, it's clear that $[\Pi(\bar{x}), \Pi(\bar{y})] = [\Pi(\bar{x}), \Pi(\bar{y})^+] = 0$. What about:

$$[\phi(\bar{x}), \Pi(\bar{y})] = \iint \frac{d^3p\, d^3p'}{(2\pi)^6} \sqrt{\frac{\omega_{p'}}{\omega_p}} (-i) \left[A_{\bar{p}} + B^+_{-\bar{p}},\ A_{\bar{p}} - B^+_{-\bar{p}} \right] e^{i\bar{p}\cdot\bar{x} + i\bar{p}'\cdot\bar{y}}$$

$$= \frac{(-i)}{(2\pi)^6} \iint_{p\, p'} d^3p\, d^3p' \sqrt{\frac{\omega_{p'}}{\omega_p}} \left(\text{zero} \right) e^{\sim}$$

$$= 0$$

$$[\phi(\bar{x}), \Pi^*(\bar{y})] = \frac{(i)}{(2\pi)^6} \iint_{p\, p'} d^3p\, d^3p' \sqrt{\frac{\omega_{p'}}{\omega_p}} \left[A_{\bar{p}} + B^+_{-\bar{p}},\ A^+_{\bar{p}'} - B_{-\bar{p}'} \right] e^{i\bar{p}\cdot\bar{x} - i\bar{p}'\cdot\bar{y}}$$

$$= \frac{(i)}{(2\pi)^6} \int_p \int_{p'} d^3p\, d^3p' \sqrt{\frac{\omega_{p'}}{\omega_p}} \left([A_{\bar{p}} + A^+_{\bar{p}'}] - [B^+_{-p}, B_{-p'}] \right) e^{i\bar{p}\cdot\bar{x} - i\bar{p}'\cdot\bar{y}}$$

$$= \frac{(i)}{2(2\pi)^3} \int_p \int_{p'} d^3p\, d^3p' \sqrt{\frac{\omega_{p'}}{\omega_p}} \left(\delta(p - \bar{p}') + \delta(\bar{p} - \bar{p}') \right) e^{i(\bar{p}\cdot\bar{x} - \bar{p}'\cdot\bar{y})}$$

should not have * ; see your p2

$$[\phi(\bar{x}), \pi^*(\bar{y})] = \frac{(i)}{(2\pi)^3}\int_p d^3p \, e^{i\bar{p}\cdot(\bar{x}-\bar{y})} = i\delta(\bar{x}-\bar{y})$$

Second quantization is satisfied, as it should be. Let's calculate the hamiltonian.

$$H = \int_x (\pi^*\pi + \bar{\nabla}\phi^*\cdot\bar{\nabla}\phi + m^2\phi^*\phi)\, d^3x$$

$$\int_x \pi^*\pi \, d^3x = \iiint_{x\,p\,p'} \frac{d^3p\, d^3p'\, d^3x}{(2\pi)^6} (i)(-i)\sqrt{\omega_p\omega_{p'}} \left(A^\dagger_{\bar{p}} - B_{-\bar{p}}\right)\left(A_{\bar{p}'} - B^\dagger_{-\bar{p}'}\right) e^{-i\bar{p}\cdot\bar{x} + i\bar{p}'\cdot\bar{x}}$$

$$= \iiint_{x\,p\,p'} \frac{d^3p\, d^3p'\, d^3x}{(2\pi)^6} \sqrt{\omega_p\omega_{p'}} \left(A^\dagger_{\bar{p}}A_{\bar{p}'} - A^\dagger_{\bar{p}}B^\dagger_{-\bar{p}'} - B_{-\bar{p}}A_{\bar{p}'} + B_{-\bar{p}}B^\dagger_{-\bar{p}'}\right) e^{i(-\bar{p}\cdot\bar{x} + \bar{p}'\cdot\bar{x})}$$

$$\int_x \bar{\nabla}\phi^*\cdot\bar{\nabla}\phi \, d^3x = \iiint_{x\,p\,p'} \frac{d^3p\, d^3p'\, d^3x}{(2\pi)^6} \frac{1}{\sqrt{\omega_p\omega_{p'}}} (-i\bar{p}\cdot i\bar{p}')\left(A^\dagger_{\bar{p}} + B_{-\bar{p}}\right)\left(A_{\bar{p}'} + B^\dagger_{-\bar{p}'}\right) e^{-i\bar{p}\cdot x + i\bar{p}'\cdot\bar{x}}$$

$$= \iiint_{x\,p\,p'} \frac{d^3p\, d^3p'\, d^3x}{(2\pi)^6} \frac{(\bar{p}\cdot\bar{p}')}{\sqrt{\omega_p\omega_{p'}}} \left(A^\dagger_{\bar{p}}A_{\bar{p}'} + A^\dagger_{\bar{p}}B^\dagger_{-\bar{p}'} + B_{-\bar{p}}A_{\bar{p}'} + B_{-\bar{p}}B^\dagger_{-\bar{p}'}\right) e^{i(\bar{p}'\cdot\bar{x} - \bar{p}\cdot\bar{x})}$$

$$\int_x m^2\phi^*\phi \, d^3x = \iiint_{x\,p\,p'} \frac{d^3p\, d^3p'\, d^3x}{(2\pi)^6} \frac{m^2}{\sqrt{\omega_p\omega_{p'}}} \left(A^\dagger_{\bar{p}}A_{\bar{p}'} + A^\dagger_{\bar{p}}B^\dagger_{-\bar{p}} + B_{-\bar{p}'}A_{\bar{p}'} + B_{\bar{p}}B^\dagger_{\bar{p}'}\right) e^{i(\bar{p}'\cdot\bar{x} - \bar{p}\cdot\bar{x})}$$

all exponents: $e^{i(\bar{p}' - \bar{p})\cdot\bar{x}} \rightarrow (2\pi)^3\delta(\bar{p} - \bar{p}')$

$\bar{p}' \rightarrow \bar{p}$

$$\int_x \pi^* \pi \, d^3x = \int_p \int_{p'} \frac{d^3p \, d^3p'}{(2\pi)^3} \sqrt{\omega_p \omega_{p'}} \, \delta(\bar{p}-\bar{p}') \left[A^\dagger_{\bar{p}} A_{\bar{p}'} - A^\dagger_{\bar{p}} B^\dagger_{-\bar{p}'} - B_{-\bar{p}} A_{\bar{p}'} + B_{-\bar{p}} B^\dagger_{-\bar{p}'} \right]$$

$$= \int_p \frac{d^3p}{(2\pi)^3} \omega_p \left[A^\dagger_{\bar{p}} A_{\bar{p}} - A^\dagger_{\bar{p}} B^\dagger_{-\bar{p}} + B_{-\bar{p}} A_{\bar{p}} - B_{\bar{p}} B^\dagger_{\bar{p}} \right]$$

$-p\,-p \to p\,p$

$$\int_x \bar{\nabla}\phi^* \cdot \bar{\nabla}\phi \, d^3x = \int_p \frac{d^3p}{(2\pi)^3} \frac{p^2}{\omega_p} \left[A^\dagger_{\bar{p}} A_{\bar{p}} + A^\dagger_{\bar{p}} B^\dagger_{-\bar{p}} + B_{-\bar{p}} A_{\bar{p}} + B_{\bar{p}} B^\dagger_{\bar{p}} \right]$$

$$\int_x m^2 \phi^* \phi \, d^3x = \int_p \frac{d^3p}{(2\pi)^3} \frac{m^2}{\omega_p} \left[A^\dagger_{\bar{p}} A_{\bar{p}} + A^\dagger_{\bar{p}} B^\dagger_{-\bar{p}} + B_{-\bar{p}} A_{\bar{p}} + B_{\bar{p}} B^\dagger_{\bar{p}} \right]$$

$$H = \int_p \frac{d^3p}{(2\pi)^3} \omega_p \left[A^\dagger_{\bar{p}} A_{\bar{p}} + B_{\bar{p}} B^\dagger_{\bar{p}} \right]$$

but since $[B_{\bar{p}}, B^\dagger_{\bar{p}'}] = (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{p}')$, we have:

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p \left[A^\dagger_{\bar{p}} A_{\bar{p}} - B^\dagger_p B_p \right]$$

where did – come from

$BB^\dagger$
$= BB^\dagger - B^\dagger B + B^\dagger B$
$= [B, B^\dagger] + B^\dagger B$

I threw away the divergent part of the hamiltonian

The Hamiltonian is made up of 2 independent pieces — they commute, so in fact they must represent <u>different</u> particles. This 2-fields-in-1 generates negative probability density, so we should reinterpret the oscillators as representing positive/negative charge annhilation/creation operators.

c) $Q = \frac{i}{2}\int_x (\phi^* \pi^* - \pi\phi) d^3x$

$$= \frac{i}{2}\int_x\int_p\int_{p'} \frac{d^3p\, d^3p'\, d^3x}{(2\pi)^6}\sqrt{\frac{\omega_{p'}}{\omega_p}}\; i\left(A^\dagger_{\bar p} + B_{-\bar p}\right)\left(A^\dagger_{\bar p'} - B_{-\bar p'}\right)e^{-i(\bar p + \bar p')\cdot \bar x}$$

$$- \frac{i}{2}\int_x\int_p\int_{p'} \frac{d^3p\, d^3p'\, d^3x}{(2\pi)^6}\sqrt{\frac{\omega_{p'}}{\omega_p}}\,(-i)\left(A_{\bar p} - B^\dagger_{-\bar p}\right)\left(A_{\bar p'} + B^\dagger_{-\bar p'}\right)e^{i(\bar p + \bar p')\cdot \bar x}$$

$$= -\frac{1}{2}\int_p\int_{p'} \frac{d^3p\, d^3p'}{(2\pi)^3}\sqrt{\frac{\omega_{p'}}{\omega_p}}\,\delta(\bar p + \bar p')\left[A^\dagger_{\bar p} A^\dagger_{\bar p'} - A^\dagger_{\bar p} B_{-\bar p'} + B_{-\bar p} A^\dagger_{\bar p'} - B_{-\bar p} B_{-\bar p'}\right]$$

$$- \frac{1}{2}\int_p\int_{p'} \frac{d^3p\, d^3p'}{(2\pi)^3}\sqrt{\frac{\omega_{p'}}{\omega_p}}\,\delta^3(\bar p + \bar p')\left[A_{\bar p} A_{-\bar p} + A_{\bar p} B^\dagger_{-\bar p'} - B^\dagger_{-\bar p} A_{\bar p'} - B^\dagger_{-\bar p} B^\dagger_{-\bar p'}\right]$$

$$= -\frac{1}{2}\int_p \frac{d^3p}{(2\pi)^3}\Big[A^\dagger_{\bar p} A^\dagger_{-\bar p} - A^\dagger_{\bar p} B_{\bar p} + B_{-\bar p} A^\dagger_{-\bar p} - B_{-\bar p} B_{\bar p} + A_{\bar p} A_{-\bar p}$$

$$+ A_{\bar p} B^\dagger_{\bar p} - B^\dagger_{-\bar p} A_{-\bar p} - B^\dagger_{-\bar p} B^\dagger_{\bar p}\Big]$$

$$= -\frac{1}{2}\int_p \frac{d^3p}{(2\pi)^3}\Big[A^\dagger_{\bar p} A^\dagger_{-\bar p} + A_{\bar p} A_{-\bar p} - B^\dagger_{-\bar p} B^\dagger_{\bar p} - B_{-\bar p} B_{\bar p}$$

$$+ \underbrace{[B_{-\bar p}, A^\dagger]}_{0} + \underbrace{[A_{\bar p}, B^\dagger_{\bar p}]}_{0}\Big]$$

$$= -\frac{1}{4}\int_p \frac{d^3p}{(2\pi)^3}\Big[(a^\dagger_{\bar p} - i b^\dagger_{\bar p})(a^\dagger_{-\bar p} - i b^\dagger_{-\bar p}) + (a_{\bar p} + i b_{\bar p})(a_{-\bar p} + i b_{-\bar p})$$

$$- (a^\dagger_{-p} + i b^\dagger_{-\bar p})(a^\dagger_p + i b^\dagger_{\bar p}) - (a_{-\bar p} - i b$$