Perturbative Expansion of Correlation Functions

A useful quantity to calculate is the n-point correlation (or Green) function, given by $\langle\Omega|T\,\phi(x_1)\cdots\phi(x_N)|\Omega\rangle$. By sneaky + ingeneous methods, we can approximate:

$$\boxed{\langle\Omega|T\phi(x_1)\cdots\phi(x_N)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\,\phi_I(x_1)\cdots\phi_I(x_N)\,e^{-i\int_{-T}^{T}H_I(t)\,dt}|0\rangle}{\langle 0|T\phi_I(x_1)\cdots\phi_I(x_N)|0\rangle}}$$

where $H = H_0 + H_I = \begin{cases} H_{\text{Klein Gordon}} + \frac{\lambda}{4!}\int\phi^4(x)\,d^3x & \phi^4 \text{ theory} \\ H_{\text{Dirac}} + H_{\text{Klein Gordon}} + \int g\bar{\Psi}\Psi\phi\,d^3x & \text{yukawa theory} \end{cases}$

In free field theory, the 2-point function becomes the Feynman propagator $D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle$

Wick's Theorem (All ϕ's are ϕ_I)

Define the contraction (a c-number) as:

$$\overbrace{\phi(x)\phi(y)} \equiv D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \begin{cases}[\phi^+(x),\phi^-(y)] & x^0 > y^0 \\ [\phi^+(y),\phi^-(x)] & x^0 < y^0\end{cases}$$

where $\phi(x) = \phi^+(x) + \phi^-(x)$ and $\phi^+(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_p e^{-ip\cdot x} \iff \phi^+|0\rangle = 0$

$$\phi^-(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_p^\dagger e^{ip\cdot x} \iff \langle 0|\phi^- = 0$$

Define normal order to mean all the dagger terms on the left.

$$\boxed{T\{\phi(x_1)\cdots\phi(x_N)\} = N\{\text{all contractions of } \phi(x_1)\cdots\phi(x_N)\}}$$

Important: Free field contraction of any normal ordered product is zero: $\langle 0|N\{\cdots\}|0\rangle = 0$ but $\langle 0|\text{contractions}|0\rangle = \text{contractions}$. Time ordered product of an odd # of fields is always zero.

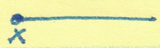
Feynman Diagrams : ϕ^4 Theory

Expand the numerator of $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$

$$= \langle 0 | T \{ \phi(x)\phi(y) + (-i)\phi(x)\phi(y) \int H_I(z_1) d^4 z_1 + \frac{(-i)^2}{2!} \phi(x)\phi(y) \int H_I(z_1) d^4 z_1 \int H_I(z_2) d^4 z_2 + \cdots + \cdots \} | 0 \rangle$$

The rules for ϕ^4 theory:

momentum space

1. For each propagator  $= D_F(x-y) = \dfrac{i}{p^2 - m^2 + i\varepsilon}$
2. For each vertex  $= (-i\lambda)\int d^4 z = -i\lambda$
3. For each external point  $= 1 = e^{-ip\cdot x}$

In momentum space, each undetermined momentum gets integrated $\int \frac{d^4 p}{(2\pi)^4}$.

4. Divide by symmetry factor

The denominator $\langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle$ = diagrams with propagators which are disconnected from external lines. For example:

In general, $\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle$ = sum of all connected diagrams with n ext points.

Amputated diagrams are when loops off propagators are removed: 

$\boxed{iM = \text{sum of all connected amputated diagrams}}$

Cross Sections and the S-Matrix

We like to compute things like $P = | \underbrace{\langle \phi_1 \phi_2 \cdots}_{\text{state of several future wave packets, one for each final state particle}} | \underbrace{\phi_A \phi_B \rangle}_{\text{Two wave packets constructed in the far past.}} |$

-or- $_{out}\langle p_1 p_2 \cdots | k_A k_B \rangle_{in} = {}_{in}\langle p_1 p_2 \cdots | S | k_A k_B \rangle_{in}$ where

$$S = 1 + iT = 1 + i(2\pi)^4 \delta^4(k_A + k_B - \textstyle\sum p_f) M(k_A, k_B \rightarrow p_f)$$

Fermionic Reduction Formula (Yukawa Theory)

$$H_{Yukawa} = H_{KG} + H_{Dirac} + \int g \bar{\Psi} \Psi \phi \, d^3x$$

Time ordering in Yukawa Theory is done a bit differently

$$T[\Psi(x)\bar{\Psi}(y)] = \begin{cases} \Psi(x)\bar{\Psi}(y) & x^0 > y^0 \\ -\bar{\Psi}(y)\Psi(x) & x^0 < y^0 \end{cases}$$

The Feynmann propagator for the Dirac field:

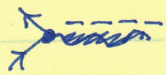
$$S_F(x-y) = \overbrace{\Psi(x)\bar{\Psi}(y)} = \langle 0 | T\,\Psi(x)\bar{\Psi}(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$$

Feynmann Rules

① Propagators

$- - \rightarrow - -$ $= \dfrac{i}{q^2 - m_\phi^2 + i\varepsilon}$ scalar propagator

$\longrightarrow$ $= \dfrac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$ Fermionic propagator

② Vertex  $-ig$

③ External fields:

U^s incoming f $\qquad$ V^s outgoing $\bar{f}$

$\bar{U}^s$ outgoing f $\qquad$ $\bar{V}^s$ incoming $\bar{f}$

④ Conserve momentum at each vertex

⑤ Integrate over each undetermined loop momentum

⑥ Figure out overall sign of diagram

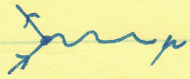
notes: No symmetry factor; the $\frac{1}{n!}$ factor from Taylor's expansion is always cancelled. Always move backwards against fermionic flow.

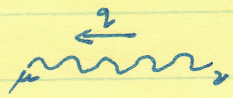
Quantum Electrodynamics

Replace scalar particle ϕ from Yukawa Theory by vector particle A_μ and change the interaction Hamiltonian to:

$$H_I = \int e \bar{\Psi} \gamma^\mu \Psi A_\mu \, d^3x$$

Feynman Rules:

① Vertex  $= -ie\gamma^\mu$

② Photon propagator  $-\dfrac{i g_{\mu\nu}}{q^2 + i\varepsilon}$

③ External photon lines

ingoing $A_\mu = \varepsilon_\mu(p)$

outgoing $A_\mu = \varepsilon^*_\mu(p)$

polarization vectors of initial or final state photon

④ External fermionic lines are the same as with Yukawa Theory.

Each component of A obeys the Klein-Gordon equation (with $m=0$)

$$A_\mu = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=0}^{3} \left[a_p^r \varepsilon_\mu^r(p) e^{-ip\cdot x} + a_p^{r\dagger} \varepsilon_\mu^{r*}(p) e^{ip\cdot x} \right]$$