Let Ω be the vacuum state of the universe (for a non-trivial H). A real vacuum. A physical vacuum, as opposed to $|0\rangle$. So we have a $|\phi\rangle, \hat{H}$ and Ω and we'd like to calculate probabilities, scattering matrices, etc. There exists a theorem: the LSZ reduction theorem if you're interested in an in-state at $t=-\infty$ and an out-state at $t=\infty$

$$\langle in, t=-\infty \mid out, t=+\infty\rangle \sim \langle \Omega | \hat{T}_{\phi\ldots\phi} | \Omega\rangle$$

We want to calculate this kind of vacuum expectation value. How do we calculate it? Use PT: free particles + interaction hamiltonian:

$\phi, H, |\Omega\rangle \longrightarrow$ H_0 free hamiltonian

H_I interaction hamiltonian

$|\phi\rangle$ interaction field

$|0\rangle$ vacuum for H_0

$$U(t,t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}$$

$$= \hat{T} \exp\{-i \int_{t'}^{t} H_I(t'')\,dt''\}$$

Start with this stuff which we can calculate and get a perturbative expression for $\langle \Omega | T_{\phi\ldots\phi} | \Omega\rangle$. We need to write the physical vacuum in terms of free states $|0\rangle$.

Trick: Write vacuum as a sum of states of free field vacuum $|0\rangle$.

$$|0\rangle = \sum |n\rangle\langle n|0\rangle \quad \leftarrow$$ sum of states over full Hamiltonian. We don't know what they are but we know they exist.

Look at $e^{-iHT}|0\rangle = \sum e^{-iE_nT}|n\rangle\langle n|0\rangle$ T is a large time, not time ordering

Now take the limit as $T \to \infty(1-i\varepsilon)$

$$\lim e^{-iHT}|0\rangle = \lim_{T\to\infty(1-i\varepsilon)} \sum e^{-iE_nT} e^{-\varepsilon E_nT} |n\rangle\langle n|0\rangle \quad \leftarrow$$ all terms except for small n vanish

$$= \lim_{T\to\infty(1-i\varepsilon)} e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle \quad \leftarrow$$ state with lowest eigenvalue. Hidden assumption is that $|\Omega\rangle$ is unique. If not, the whole argument breaks down.

Solve this for $|\Omega\rangle$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left[e^{-iE_0T}\langle\Omega|0\rangle\right]^{-1} e^{-iHT}|0\rangle \quad \leftarrow$$ Physical vacuum in terms of the free field vacuum.

By same type of argument we can calculate

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|e^{-iHT}\left(e^{-iE_0T}\langle 0|\Omega\rangle\right)^{-1}$$

$$|\Omega\rangle = \text{limit}\,(\quad)^{-1} e^{-iHT} e^{+iH_0T} |0\rangle$$

↰ sticking in factor of 1 since $H_0|0\rangle = |0\rangle$.

$$u(t,t') = e^{-iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}$$

t_0 was some arbitrary starting point for the interaction picture. WLOG, chose $t_0=0$.

Now if $t=0$, then $U(0,0) = \cdots$

$$\Rightarrow |\Omega\rangle = \text{limit}\,(\quad)^{-1} U(0,-T)|0\rangle$$

propagate infinite past to 0

$$\langle\Omega| = \text{limit}\,\left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1}\langle 0|U(T,0)$$

propagate $|0\rangle$ at $t=0$ to $t=\infty$

For simplicity: $x_0 > y_0 > 0$.

Expectations of time ordered ~~pairs~~ products

$$\langle\Omega|\hat{T}\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle\right)^{-1} \overset{\langle\Omega|}{\langle 0|U(T,0)}\, \underbrace{U^{\dagger}(x^0,0)\phi_I(x)U(x^0,0)}_{\text{definition of interaction picture}}$$

$$U^{\dagger}(y^0,0)\phi_I(y)U(y^0,0)\, \overset{|\Omega\rangle}{U(0,-T)|0\rangle}$$

We didn't write a time ordering since we assumed $x^0 > y^0$. However, we can stick it in. $U^{\dagger}(T,0) = U(0,T)$

$$\langle\Omega|\hat{T}\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} (\;)^{-1}\langle 0|T\{\underbrace{U(T,0)U(0,x^0)}_{U(T,x^0)}\phi_I(x)\underbrace{U(x_0,0)U(0,y^0)}_{U(x_0,y_0)}\phi_I(y)\underbrace{U(y_0,0)U(0,-T)}_{U(y_0,-T)}\}|0\rangle$$

$$= \text{limit}\,(\;)^{-1}\langle 0|T\{U(T,x^0)\,\phi_I(x)\,U(x_0,y_0)\,\phi_I(y)\,U(y_0,-T)\}|0\rangle$$

Because of $\hat{T}$ we can change the order of these things.

$$\langle\Omega|\hat{T}\phi(x)\phi(y)|\Omega\rangle = \text{limit}\ (\ )^{-1}\langle 0|T\{\phi_I(x)\phi_I(y)\underbrace{U(T,x_0)U(x_0,y_0)U(y_0,-T)}_{U(T,-T)}\}|0\rangle$$

$$= \text{limit}\ (\ )^{-1}\langle 0|T\{\phi_I(x)\phi_I(y)e^{-i\int_{-T}^{T}H_I(t)dt}\}|0\rangle$$

Expand exponent in terms of infinite sum of integrals

$$= \langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$$

$$\langle\Omega|\Omega\rangle = \text{limit}\ (\ )^{-1}\langle 0|Te^{-i\int H_I dt}|0\rangle$$

$$\Rightarrow\ (\ )^{-1} = \left(\langle 0|Te^{-i\int_{-T}^{T}H_I dt}|0\rangle\right)^{-1}$$

- we did this for 2 fields. if we had done this for 3 fields nothing would've changed
- we did this for $x_0 > y_0$. If $x_0 < y_0$ we would've gotten the same result.

Put these together and get:

$$\boxed{\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_I(x_1)\cdots\phi_I(x_n)\}e^{-i\int_{-\infty}^{\infty}H_I(t)dt}|0\rangle}{\langle 0|Te^{-i\int H_I(t)dt}|0\rangle}}$$

<u>All</u> PT starts with this expression and expands the $e^{-i\int H dt}$.

<u>nice physics</u> ↑

boring math ↓

<u>How do we calculate these time ordered products?</u>

$$\langle 0|T\phi_I(x)\cdots\phi_I(x_n)|0\rangle \leftarrow$$ rewrite this in terms of commutators. Move creation to the left and annihilation to the right.

We're going to end up with Wick's Theorem.

Consider 2 fields (1 field gives zero): $\phi_I(x) = \phi_I^+ + \phi_I^-(x)$

ϕ_I^+: involves only a_p $\qquad$ ϕ_I^-: involves only a_p^+

From now on we'll drop the "I" but it's understood that everything is interaction picture.

$$\phi^+|0\rangle = 0 \qquad \text{and} \qquad \langle 0|\phi^- = 0$$

Assume $x^0 > y^0$. Then:

$$\hat{T}\phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y)$$

Now define the normal ordering operator: $\hat{N}$ or $:(\cdots):$ such that

$\hat{N}(\text{Fields})$ = Fields with all positive freq parts ϕ^+ on right

So, eg:

$$\hat{T}\phi(x)\phi(y) = \begin{cases} \hat{N}\phi(x)\phi(y) + [\phi^+(x), \phi^-(y)] & \text{if } x^0 > y^0 \\ \hat{N}\phi(y)\phi(x) + [\phi^+(y), \phi^-(x)] & \text{if } y^0 > x^0 \end{cases}$$

Define a contraction:

$$\overline{\phi(x)\phi(y)} = \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & y^0 > x^0 \end{cases}$$

Then $T\phi(x)\phi(y) = N\phi(x)\phi(y) + \overline{\phi(x)\phi(y)}$. Take the vacuum expect. value:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|\overline{\phi(x)\phi(y)}|0\rangle$$

This tells you what a contraction is! $\quad \overline{\phi(x)\phi(y)} = D_F(x-y)$

Wicks Theorem

$$\langle 0|T\phi_1, \cdots, \phi_n|0\rangle = \langle 0|\hat{N}\phi_1\cdots\phi_n|0\rangle + \langle 0|\hat{N}\sum_{\substack{\text{all contractions}\\ \text{of 2 or more}\\ \text{fields}}}|0\rangle$$

so for 4 fields: $\overline{\phi_1\phi_2}\phi_3\phi_4 + \overline{\phi_1\phi_2\phi_3}\phi_4 + \cdots + \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \overline{\phi_1\phi_2\phi_3}\phi_4 + \cdots$

For odd number of fields you get zero.