Dirac Propagator

We need to compute 2 things first:

① vacuum expectation: $\langle 0|\psi_a(x)\bar{\psi}_b(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u_a^s(p)\bar{u}_b^s}_{\not{p}+m} e^{-ip\cdot x}e^{ip\cdot y}$

$$\psi_a(x) \sim \int d^3p\,(au+b^\dagger \bar{v}) \qquad \bar{\psi}_b(x) \sim \int d^3p\,(a^\dagger u + bv)$$

$$\langle 0|\psi_a(x)\bar{\psi}_b(x)|0\rangle = (i\not{\partial}_x + m)\underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}}_{D(x-y)}$$

② $\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s v_a^s(p)\bar{v}_b^s(p)}_{\not{p}-m} e^{-ip\cdot(x-y)}$

$$= -(i\not{\partial}_x+m)D(y-x)$$

We're interested in computing a Dirac propagator. There are different ways of computing one (ie- retarded, Feynman, advanced). Consider:

$$S_R(x-y) = (i\not{\partial}_x+m)\underbrace{D_R(x-y)}_{\text{propagator for scalar field case } = \theta(x^0-y^0)[D(x-y)-D(y-x)]}$$

$$= \theta(x^0-y^0)[(i\not{\partial}_x+m)D(x-y) - (i\not{\partial}_x+m)D(y-x)] + i\gamma^0\partial_{x^0}\theta(x^0-y^0)\underbrace{[D(x-y)-D(y-x)]}_{\text{if times are equal this is equal to zero.}}$$

$$= \theta(x^0-y^0)\langle 0|\{\psi_a(x),\bar{\psi}_b(y)\}|0\rangle$$

Now we have to check that $S_R(x-y)$ obeys the equation of motion.

$$(i\not{\partial}_x-m)S_R(x-y) = (-\partial_x^2-m^2)D_R(x-y)\,\mathbb{1}_{4\times4} = (i\not{\partial}_x-m)(i\not{\partial}_x+m)D_R(x-y)\,\mathbb{1}_{4\times4}$$

$$= i\delta^4(x-y)\,\mathbb{1}_{4\times4}$$ recall $\forall$ slash vectors $\not{p}\not{p}=p^2$. We used that for $\not{\partial}_x$.

Most general form is obtained by writing S_R in Fourier transform form:

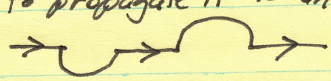
$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \hat{S}(p) e^{-ip\cdot(x-y)}$$

$$\int \frac{d^4p}{(2\pi)^4} (\not{p}-m)\hat{S}(p) e^{-ip\cdot(x-y)} = i\delta^4(x-y)\,\mathbb{1}_{4\times4} \implies \boxed{\hat{S}(p) = \frac{i}{\not{p}-m}}$$

Any S has to take the form

$$\hat{S}(p) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p}-m} e^{-ip\cdot(x-y)} \quad \text{but} \quad \frac{i}{\not{p}-m} = \frac{i}{\not{p}-m}\frac{\not{p}+m}{\not{p}+m} = \frac{i(\not{p}+m)}{p^2-m^2}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} (\not{p}+m) e^{-ip(x-y)}$$

This propagator propagates a particle from one place to another. If we have a field at some location and want to propagate it to another place, this is how we do it. We'll we the Feynman prescription  for the p^0 plane.

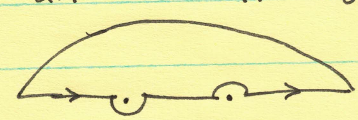
$$S = \begin{cases} \langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar{\Psi}(y)\Psi(x)|0\rangle & x^0 < y^0 \end{cases} \equiv \langle 0|T\{\Psi(x), \bar{\Psi}(y)\}|0\rangle$$

extra minus sign related to the anticommutation game we play for fermionic fields. This is different from the scalar field!

$$\int \frac{dp_0}{(2\pi)} \frac{d^3p}{(2\pi)^3} \frac{i(\not{p}+m)e^{-ip_0(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{(p_0-E_p+i\tilde{\varepsilon})(p_0+E_p-i\tilde{\varepsilon})\}} \longrightarrow p_0^2 - E_p^2 + \underbrace{i\tilde{\varepsilon}(\cancel{p_0}+E_p) - i\tilde{\varepsilon}(\cancel{p_0}-E_p)}_{=2i\tilde{\varepsilon}E_p = i\varepsilon} + O(\varepsilon^2)$$

pole at $-E_p$ is shifted down. pole at $+E_p$ is shifted up.

Let's evaluate S for the case $x^0 < y^0$ and see what happens. Since $x^0<y^0$ we close upwards $p^0 \to iR$



we'll pick up the pole at $-E_p$.

$$S = \frac{2\pi i}{2\pi}\, i \int \frac{d^3p}{(2\pi)^3} \frac{(-E_p\gamma^0 - \vec{p}\vec{\gamma} + m)}{-2E_p} e^{+iE_p(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y})}$$

COV: $\vec{p} \to -\vec{p}$

$$S = -\int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{(-E_p\gamma^0 + \vec{p}\cdot\vec{\gamma} + m)}{-2E_p}}_{\frac{(\not{p}-m)}{2E_p}} \underbrace{e^{-iE_p(x^0-y^0) - i\vec{p}\cdot(\vec{x}-\vec{y})}}_{e^{-ip\cdot(x-y)}}$$

which equals $\langle 0|T\{\psi(x), \bar{\psi}(y)\}|0\rangle$ for the case $x^0 < y^0$. S is a Green function for Dirac theory. Green function $\equiv$ propagator.

The CPT Game

Charge conjugation
Parity
Time Reversal

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) = \psi'(x)$$

equivalent to $\psi'(x') = \Lambda_{\frac{1}{2}}\psi(x)$
↳ mixes up Dirac components

Parity: $\quad P^0 : (t, x) \rightarrow (t, -\bar{x})$

Time Reversal: $\quad T' : (t, x) \rightarrow (-t, \bar{x})$

Discontinuous transformations! They cannot be developed from infintessmal changes of identity transformation

$\left\{ \begin{matrix} \text{Proper Lorentz} \\ \text{transformation} \end{matrix} \right\}$ + Parity + time reversal = 4 disconnected sets

---

<u>parity</u>

What happens to momentum and spin under parity?

$\vec{p} \rightarrow -\vec{p}$ sign reversal
$\bar{s} \rightarrow \bar{s}$ stays same

$$P a^{st}_{\vec{p}} |0\rangle \propto a^{st}_{-\vec{p}} |0\rangle$$

$$P a^{st}_{p} P P^{-1} |0\rangle = \eta_a^* a^{st}_{-p} |0\rangle$$

When we write P we mean $U(\Lambda)$ for P as P itself. Also, $P^{-1} = P$.

but $P^{-1}|\text{vacuum}\rangle = P|\text{vacuum}\rangle = |\text{vacuum}\rangle$, so:

$$P a^{st}_p P = \eta_a^* a^{st}_{-p}$$
$$P b^{st}_p P = \eta_b^* b^{st}_{-p}$$

we know this physically. if we do P twice we get:

$$\underset{1}{P^2} a_p^{st} \underset{1}{P^2} = \eta_a^* \underbrace{P a_{-p}^{st} P}_{\eta_a^* a_p^{st}} \overset{so}{\Longrightarrow} (\eta_a^*)^2 = 1$$

Similarly, $(\eta_b^*)^2 = 1$. So: $\begin{cases} \eta_a^* = \pm 1 \\ \eta_b^* = \pm 1 \end{cases}$

$$P\psi(x)P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(\eta_a a_{-p}^s U^s(p) e^{-ip\cdot x} + \eta_b^* b_{-p}^{st} V^s(p) e^{ip\cdot x} \right)$$

$\underset{\text{equal}}{\overset{\text{this must}}{=}}$ $\underset{\text{some matrix}}{M}\, \psi(\Lambda^{-1}x) = \psi(\tilde{x})$ where $\Lambda^{-1}x = (x^0, -\vec{x}) = \tilde{x}$

spatially flipped location

It's not clear that the large expression equals $M\psi(\Lambda^{-1}x)$. Let's check it out. First we need some identities.

$$U(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi \\ \sqrt{\tilde{p}\cdot\sigma}\,\xi \end{pmatrix} = \gamma^0 \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,\xi \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi \end{pmatrix} = \gamma^0 U(\tilde{p})$$

Similarly $V(p) = \underset{NB}{-}\gamma^0 V(\tilde{p})$. Also useful is $\tilde{p}\cdot x = \tilde{p}\cdot\tilde{x}$

$$P\psi(x)P = \underbrace{\int \frac{d^3p}{(2\pi)^3}}_{=\frac{d^3\tilde{p}}{(2\pi)^3}} \frac{1}{\sqrt{2E_p}} \sum_s \left[\eta_a \underbrace{a_{-p}^s}_{=a_{\tilde{p}}^s} \gamma^0 U(\tilde{p}) \underbrace{e^{-ip\cdot x}}_{=e^{-i\tilde{p}\cdot x}} + \eta_b \underbrace{b_{-p}^{st}}_{=b_{\tilde{p}}^{st}} (-\gamma_0 V^s(\tilde{p})) e^{i\tilde{p}\cdot\tilde{x}} \right]$$

Now redefine $\tilde{p}$ as p.

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left[\eta_a a_p^s \gamma^0 U^s(p) e^{-ip\cdot\overset{NB}{\tilde{x}}} - \eta_b^* b_p^{st} \gamma^0 V^s(p) e^{+ip\cdot\overset{NB}{\tilde{x}}} \right]$$

If we ~~those~~ choose $\eta_a = 1$ (could've been -1) then $\eta_b = -1$ in which case $M = \gamma^0$. Thus ψ is a representation of parity only if η_a, η_b have opposite signs. Therefore, a Dirac particle and Dirac anti-particle have opposite ~~(?)~~ intrinsic parity.

$P \underbrace{a_{p_1}^{\dagger} b_{p_2}^{\dagger}}_{\text{put together with zero angular momentum}} |0\rangle$ $\qquad \forall$ Lorentz transformations: $\Lambda_{\frac{1}{2}} \gamma^\mu \Lambda_{\frac{1}{2}}^{-1} = (\Lambda^{-1})^\mu{}_\nu \gamma^\nu$