How to quantize the Dirac Field

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi = \Psi^+ \rho(i\partial_0 \gamma^0 + i\partial_i \gamma^i - m)\Psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = i\Psi^+ = \Pi_\Psi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi^+)} = 0$$ For this form of the Lagrangian

$$H = \int d^3x\,(\Pi_\Psi \partial_0 \Psi - \mathcal{L}) = -\int d^3x\, \underbrace{\Psi^+\gamma^0}_{\bar\Psi}(\underbrace{i\partial_i\gamma^i}_{i\vec\nabla\cdot\vec\gamma} - m)\Psi$$

If we let $\vec\alpha = \gamma^0\vec\gamma$ and $\beta = \gamma^0$ then $H = \int d^3x\, \Psi^+ h_{dirac} \Psi$

where $h_{dirac} = -i\vec\alpha\cdot\vec\nabla + m\beta$

How to quantize?

Try $[\Psi_a(x), \underbrace{\Pi_{\Psi_b}(y)}_{i\Psi^+}]_{x^0=y^0} = i\delta^3(\vec x - \vec y)\delta^a_b$ exactly the same as scalar field

$\left.\begin{matrix} a = 1,2,3,4 \\ b = 1,2,3,4 \end{matrix}\right\}$ 4 component objects

However, this introduces inconsistencies - it turns out that you need to use anti commutation relations

$$\Psi_a(\vec x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{i\vec p\cdot\vec x} \sum \Big(a_{\vec p}\, U^s_a(\vec p) + b^s_{-\vec p}\, V^s_a(-\vec p) \Big)$$

positive energy soln — negative energy soln

Expand in terms of eigenfunctions of Dirac eqn. we want to impose a comm relation between a + b.

$$[a^r_{\vec p}, a^{s+}_{\vec q}] = [b^r_{\vec p}, b^s_{\vec q}] = (2\pi)^3\delta(\vec p - \vec q)\,\delta^r_s$$ ← this is what we'd expect.

But if we take $\Psi(\vec x)$ and Π and try to compute the other commutation $[\Psi_a, \Pi_a]$

Modulo factor of i:

$$[\psi(\bar{x}), \psi^\dagger(\bar{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{e^{i(\bar{p}\cdot\bar{x} - \bar{q}\cdot\bar{y})}}{\sqrt{2E_p 2E_q}} \sum_{r,s} \Big([a^r_{\bar{p}}, a^{s\dagger}_{\bar{q}}]\, u^r(p)\, \bar{u}^s(q)$$

$$+ [b^r_{-p}, b^{s\dagger}_{-\bar{q}}]\, v^r_a(-\bar{p})\, \bar{v}^s_b(-\bar{q}) \Big) \gamma^0_{b'b}$$

recall

$\bar{u}\gamma^0 = u^\dagger \underbrace{\gamma^0\gamma^0}_{1}$

assume $[a, b] = 0$. Plug in assumption for commutation

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \sum_r \left(u^r(\bar{p})\, \bar{u}^r(\bar{p}) + v^r(-\bar{p})\, \bar{v}^r(-\bar{p}) \right) \gamma^0$$

We wrote down identities for this kind of sum:

$$\left(\sum_r u^r(\bar{p})\, \bar{u}^r(\bar{p})\right)_{ab'} = (\not{p} + m)_{ab'} = (E_p\gamma^0 - \bar{p}\cdot\bar{\gamma} + m)_{ab'}$$

$$\left(\sum_r v^r(-\bar{p})\, \bar{v}^r(-\bar{p})\right)_{ab'} = (\tilde{\not{p}} - m)_{ab'} = (E_p\gamma^0 + \bar{p}\cdot\bar{\gamma} - m)_{ab'}$$

add: $(2E_p\gamma^0)_{ab'}$

so

$$[\psi(\bar{x}), \psi^\dagger(\bar{y})] = \cancel{2E_p\gamma^0} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\bar{p}\cdot(\bar{x}-\bar{y})}\, 2E_p \underbrace{\gamma^0_{ab'}\gamma^0_{b'b}}_{\mathbb{1}^{ab}}$$

$$= \cancel{\int \frac{d^3p}{(2\pi)^3}}\ \delta^3(\bar{x}-\bar{y})\, \mathbb{1}^{ab}$$

Compute hamiltonian

$$H = \int d^3x\, \Psi^\dagger h_{Dirac} \Psi \overset{\text{lots of}}{\underset{\text{work}}{=}} \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(a_{\bar{p}}^{s\dagger} a_{\bar{p}}^s - b_{\bar{p}}^{s\dagger} b_{\bar{p}}^s \right)$$

looks alot like spin 0 result but the minus sign is a problem

$a_p |0\rangle = 0 \qquad a_{\bar{p}}^\dagger |0\rangle = |\bar{p}\rangle$

$b_p |0\rangle = 0 \qquad b_{\bar{p}}^\dagger |0\rangle = |\bar{q}\rangle$ negative energy

So $H|q\rangle = -E_{\bar{q}} |q\rangle \longrightarrow$ we can get arbitrarily negative energies so this is bad.

However, causality is good:

$$e^{iHt} a_p^s e^{-iHt} = a_p^s e^{-iE_p t}$$
$$b_p^s \qquad\quad = b_p^s e^{+iE_p t}$$

Plug into $\Psi(\bar{x})$:

$$\Psi(\bar{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_{\bar{p}}^s U^s(\bar{p}) e^{-ip\cdot x} + b_{\bar{p}}^s V^s(\bar{p}) e^{+ip\cdot x} \right)$$

$\bar{\Psi}(\bar{y}) =$ similar

$$[\Psi_a(\bar{x}), \bar{\Psi}_b(\bar{y})] = (i\partial_x + m)_{ab} \Big(\underbrace{D(\bar{x}-\bar{y}) - D(\bar{y}-\bar{x})}_{[\phi(\bar{x}), \phi(\bar{y})]} \Big)$$

commutes if space like separated

We got 2nd quantization + causality, but the energy is troublesome. You could fool around but there will always be a problem with one of these 3 things.

$\langle 0 | [\Psi_a, \bar{\Psi}_b] | 0 \rangle = \langle 0 | \Psi_a \bar{\Psi}_b | 0 \rangle \leftarrow$ causality works but is wierd.

The solution: How to quantize Dirac field.

$$\psi(x) = \int \frac{d^3x}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a^s_{\bar{p}} u^s(\bar{p}) e^{-ip\cdot x} + b^{\dagger}_{\bar{p}} v^s(\bar{p}) e^{+ip\cdot x} \right)$$

$$\bar{\psi} = \sum_s \left(b^s_{\bar{p}} \bar{v}^s(\bar{p}) e^{-ip\cdot x} + a^{s\dagger}_{\bar{p}} \bar{u}^s(\bar{p}) e^{ip\cdot x} \right)$$

Suppose $\{\psi_a(\bar{x}), \Pi_{\psi_b}(\bar{y})\}_{x^0=y^0} = i\delta^3(\bar{x}-\bar{y})\,\delta^{ab}$

So:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(a^{s\dagger}_{\bar{p}} a^s_{\bar{p}} - b^s_{\bar{p}} b^{s\dagger}_{\bar{p}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(a^{s\dagger}_{\bar{p}} a^s_{\bar{p}} + b^{s\dagger}_{\bar{p}} b^s_{\bar{p}} - \{ b^s_{\bar{p}}, b^{s\dagger}_{\bar{p}} \} \right)$$

$$b^{s\dagger}_p b^s_p b^r_q |0\rangle = b^{s\dagger}_p \{ b^s_p, b^{r\dagger}_q \} |0\rangle \; b^{s\dagger}_{\bar{p}} b^{r\dagger}_{\bar{q}} b^s_p |0\rangle$$

$$= (2\pi)^3 \delta^3(\bar{p}-\bar{q})\,\delta^{s,r} |\bar{q}, r\rangle$$

$$|p\rangle = \sqrt{2E_{\bar{p}}}\, a^{\dagger}_{\bar{p}} |0\rangle$$
$$|q\rangle = \sqrt{2E_{\bar{p}}}\, b^{\dagger}_{\bar{r}} |0\rangle$$

$$P = \int \frac{d^3p}{(2\pi)^3} \sum_s \bar{p} \left(a^{s\dagger}_{\bar{p}} a^s_{\bar{p}} + b^{s\dagger}_{\bar{p}} b^s_{\bar{p}} \right)$$

Observables: $O(\bar{x}) = \bar{\psi}(\bar{x})\, \Gamma\, \psi(\bar{x})$

we want $[O(\bar{x}), O(y)] = 0$ for $(x-y) < 0$