Noether's Theorem

If $\phi \to \phi(x) + \alpha\Delta\phi(x)$ and Lagrangian goes into itself times a total derivative: $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu J^\mu(x)$ then the current defined by $j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu$ is conserved in the Lorentzian sense: $j_\mu j^\mu = 0$. This means there's a conserved charge $Q = \int j^0(x)\,d^3x$ and $\frac{dQ}{dt} = 0$.

This is important because it allows us to take a given Lagrangian and determine everything from it. From the Lagrangian input, everything else has to follow, like conserved quantities. The charges are the generators of all the important transformations that characterize the invariances of our theory.

$$\boxed{\text{Scalar field} = \text{spinless particle}}$$

Example: scalar field (complex scalar field)

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \qquad \phi = \text{complex scalar field}$$

We will discuss the invariance of when ϕ changes by a phase.

$$\phi \to e^{i\alpha}\phi \qquad \phi^* \to e^{-i\alpha}\phi^*$$

$$\alpha\Delta\phi = \alpha[e^{i\alpha}\phi - \phi] = \alpha[1 + i\alpha - 1]\phi = i\alpha^2\phi$$

$$\alpha\Delta\phi^* = -i\alpha^2\phi$$

$$\Longrightarrow \left.\begin{array}{l}\Delta\phi = i\phi \\ \Delta\phi^* = -i\phi\end{array}\right\} \text{infinitesimally}$$

It's very clear that $\mathcal{L}$ is invariant under this transformation since $\mathcal{L}$ depends only on $\phi^*\phi$ and $(\partial_\mu\phi^*)(\partial^\mu\phi)$. Because $\mathcal{L} \to \mathcal{L}$, it must be the case that $J^\mu \equiv 0$.

So we have as a conserved quantity:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Delta\phi^*$$

$$= (\partial^\mu \phi^*)(i\phi) + (\partial^\mu \phi)(-i\phi^*)$$

$$= i\left[(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)\right]$$

$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} =$ not so clear whether $\partial^\mu \phi^*$ or $\partial_\mu \phi^*$

how does one interpret charge from this?

What is it? It's the charge current — we will see that this Lagrangian describes a scalar field that carries charge.

<u>Example</u> Allow for translation: $x^\mu \to x^\mu + a^\mu$

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\nu \partial_\nu \phi(x) \quad \text{infinitessimally}$$

under this translation:

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu \mathcal{L}\, \delta^\mu{}_\nu$$

Hold ν fixed so we only translate in one of the 4 directions. What is the appropriate conserved current? (Note that with a^ν fixed $\mathcal{L}$ changes by a total derivative!). Here J^μ is not zero, so:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\partial_\nu \phi}_{\text{shift in field}} - \mathcal{L}\, \delta^\mu{}_\nu$$

← stress energy tensor is conserved in the sense that $\partial_\mu T^{\mu\nu} = 0$

So whenever we have a Lagrangian that's a function of some field, we'll have a conserved stress energy momentum tensor. This will tell us what the momentum and energy operators are of the system.

$$H = \int T^{00}\, d^3x \xrightarrow{\text{so}} \mathcal{H} = T^{00}$$

$$p^i = \int T^{0i} d^3x = -\int T^0{}_i\, d^3x = -\int \pi\, \partial_i \phi\, d^3x$$

We've been talking about this scalar field $\phi(x)$ — we want to think of it as a continuum analog of a coordinate variable. $\pi(x)$ is a continuum analog of a momentum. Divide space up into many boxes. 'i' indicates which box you're sitting in. Each box has associated with it a coordinate like variable $\phi(x)$ and a momentum like variable $\pi(x)$. If we use this analogy, in going from classical field theory to QM: the natural thing to do is:

$$\left.\begin{array}{l} \phi(x) \rightarrow q^i \\ \pi(x) \rightarrow p^i \end{array}\right\} \quad \begin{array}{l} [q_i, p_j] = i\delta_{ij} \\ [q_i, q_j] = [p_i, p_j] = 0 \end{array}$$

require standard QM commutation relationships.

$$\boxed{[\phi(\bar{x}), \pi(\bar{y})] = i\delta^3(\bar{x}-\bar{y})}$$

and quantization. From this and a Lagrangian, everything else follows.

Basically, we're going to accept this, see where it takes us and if the results make sense. The field + momentum commutes with itself. The 1st thing to do with this assumption is to do some mathematics. Let's go to Fourier transform space:

$$\phi(\bar{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\bar{p}\cdot\bar{x}} \phi(\bar{p},t)$$

ϕ = real

$\Rightarrow \phi^*(p) = \phi(-p)$

This ϕ should obey the equations of motion that follow from the Lagrangian — for a scalar field:

$$\left[\frac{\partial^2}{\partial t^2} - \bar{\nabla}^2 + m^2\right]\phi = 0$$

$$\left[\frac{\partial^2}{\partial t^2} + |p|^2 + m^2\right]\phi(\bar{p},t) = 0 \qquad E^2 = \omega_p^2$$

$$\left[\partial_t^2 + \omega_{\bar{p}}\right]\phi(\bar{p},t) = 0$$

This is a harmonic oscillator. For every value of $\bar{p}$ we have a HO equation.

Harmonic Oscillator $\quad H = \frac{1}{2}P^2 + \frac{1}{2}\omega^2\phi^2$

$$\phi = \frac{1}{\sqrt{2\omega}}\left[a^+ + a\right] \qquad \left[a^+, a\right] = -1$$

$$p = i\sqrt{\frac{\omega}{2}}\left[a^+ - a\right]$$

$$\Rightarrow \quad H = \omega\left[a^+ a + \frac{1}{2}\right]$$

We will solve the HO analagously to QM.

$$\left.\begin{aligned} \phi(\bar{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[a_{\bar{p}}e^{i\bar{p}\cdot\bar{x}} + a_p^+ e^{-i\bar{p}\cdot\bar{x}}\right] \\ \pi(\bar{x}) &= \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left[a_{\bar{p}}e^{i\bar{p}\cdot\bar{x}} - a_{\bar{p}}^+ e^{-i\bar{p}\cdot\bar{x}}\right] \end{aligned}\right\}$$

we still need to show that this satisfies $[\phi(\bar{x}), \pi(\bar{x})] = i\delta^3(\bar{x}-\bar{y})$

But we also need to specify a commutation relationship between a and a^+

$$\left[a_{\bar{p}}, a_{\bar{p}'}^+\right] = (2\pi)^3\delta^3(\bar{p}-\bar{p}')$$

$$\left[a_{\bar{p}}, a_{\bar{p}'}\right] = \left[a_{p'}^+, a_{p'}^+\right] = 0$$

Make a COV $\vec{p} \to -\vec{p}$ in 2nd term. (ω_p is invariant under sign change of p)

$$\phi(\bar{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\bar{p}}}} \left[a_{\bar{p}} + a^{+}_{-\bar{p}} \right] e^{i\vec{p}\cdot\vec{x}}$$

$$\pi(\bar{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left[a_{\bar{p}} - a^{+}_{-\bar{p}} \right] e^{i\vec{p}\cdot\vec{x}}$$

we still need to compute the commutator and see if we get the right result.

$$[\phi(\bar{x}), \pi(\bar{x})] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(-\frac{i}{2}\right) \sqrt{\frac{\omega_{p'}}{\omega_p}} \underbrace{\left\{ [a^{+}_{-\bar{p}}, a_{\bar{p}'}] - [a_{\bar{p}}, a^{+}_{-\bar{p}'}] \right\}}_{-2(2\pi)^3 \delta^3(\vec{p}+\vec{p}')} e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{y})}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, i e^{i\vec{p}\cdot(\bar{x}-\bar{y})}$$

$$= i\, \delta^3(\bar{x}-\bar{y}) \quad \checkmark\checkmark$$

where we used p' for π
p for ϕ

Now come back to $p^i = \int T^{0i} d^3x$ For $\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2$

$$H = \int d^3x \left[\frac{1}{2}\pi^2(\bar{x}) + \frac{1}{2}(\bar{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right]$$

$$= \int d^3x \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \left[\cdots \overset{\text{a big}}{(a_p - a^{+}_{\bar{p}})} \overset{\text{fat mess}}{(a_{\bar{p}'} - a^{+}_{\bar{p}'})} \cdots \right]$$

$$\overset{\text{HW}}{=} \int \frac{d^3p}{(2\pi)^3} \omega_{\bar{p}} \left\{ a^{+}_{\bar{p}} a_{\bar{p}} + \frac{1}{2}[a_{\bar{p}}, a^{+}_{\bar{p}}] \right\}$$

produces ∞

There's a zero point energy for a HO – and we're integrating them over all space – an infinite number of oscillators. But we would never measure this energy – we would measure the excitations relative to the zero point (vacuum) energy.

We want to measure the energy of excitations wrt vacuum - phonons for a lattice so we measure energy of coherent motion. We define groundstate by requiring $\forall\ \vec{p}$:

$$a_{\vec{p}}\,|0\rangle = 0 \qquad a_{\vec{p}}^{+}\,|0\rangle \propto |p\rangle$$

Fock space analagous to set of excited states of system.

$$H|\vec{p}\rangle \overset{\text{modulo normalization}}{=} H a_p^{+}|0\rangle = \{[H, a_p^{+}] - a_p^{+}H\}|0\rangle$$

zero since a_p annhilates the groundstate when you ignore the infinite term in H.

$$= [H, a_p^{+}]|0\rangle$$

Now lets compute the commutator $[H, a_p^{+}]$.

$$[H, a_p^{+}] = \int \frac{d^3p'}{(2\pi)^3}\, \omega_{p'} [a_{p'}^{+} a_{p'} a_{p}^{+} - a_p^{+} a_{p'}^{+} a_{p'}]|0\rangle$$

$$= \int \frac{d^3p'}{(2\pi)^3}\, \omega_{p'} \{a_{p'}^{+} a_{p'} a_{p}^{+} - a_{p'}^{+} a_p^{+} a_{p'}\}|0\rangle$$

a, a^{+} anticommute

$$= \int \frac{d^3p'}{(2\pi)^3}\, a_{p'} a_{p'}^{+} [a_p, a_p^{+}]|0\rangle$$

$$= \omega_p a_p^{+}|0\rangle$$

$$= \omega_p |\vec{p}\rangle$$

so we verified that $H|p\rangle = \sqrt{p^2+m^2}\,|\vec{p}\rangle$

HW

Substitute π, ϕ in $-\int \pi \partial_i \phi \, d^3x$ and verify

$$\vec{p} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \, a_p^\dagger a_p$$ + no infinite constant

clever change of variables here.