

1 Kirchhoff's Diffraction Theory

The basic idea of the Huygens-Fresnel principle is that light disturbance at a point P arises from the superposition of secondary waves that proceed from a surface situated between P and the light source. It was Kirchhoff who first put the Huygens-Fresnel principle on a mathematical basis. The idea is to use Green's Theorem to express components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ at P as a surface integral of the fields and their derivatives about an arbitrary surface surrounding P .

The time dependent wave equation

$$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

must be satisfied by each component of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$. Here, f represents some component of the electric or magnetic fields.

1.1 Monochromatic Light

First consider the case where f is a component of monochromatic light (one ω):

$$f(x, y, z, t) = f_o(x, y, z) e^{-i\omega t}$$

Since f is a solution to the wave equation, its spatial component, f_o , must satisfy¹ the Helmholtz equation:²

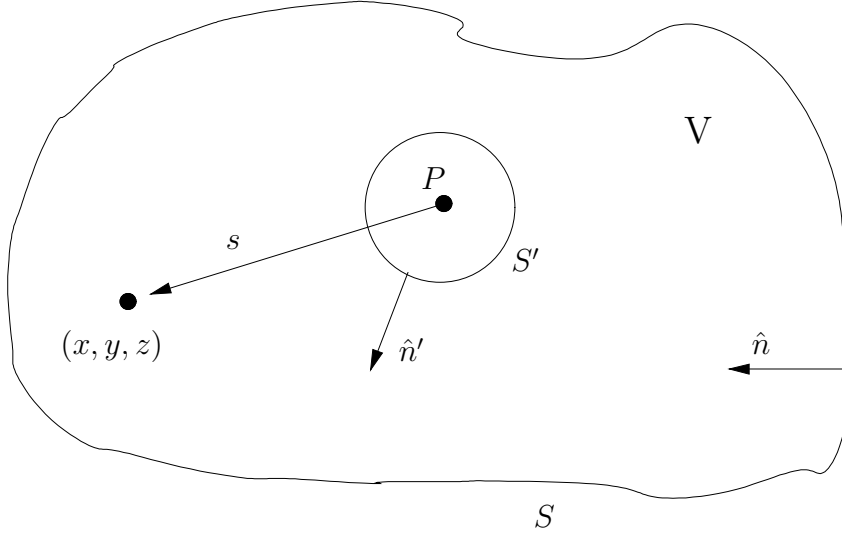
$$(\nabla^2 + k^2) f_o = 0 \quad \text{where} \quad k = \frac{\omega}{c}$$

Let V be a volume bounded by a closed surface S , and let P be a point within S . Assume that f_o is continuous and has continuous first and second order partials. Let $g(x, y, z, t)$ be *any* other solution to the wave equation with the same boundary conditions. Then the spatial components of f and g , f_o and g_o , must both satisfy Green's Theorem:³

¹You can show this by plugging f into the wave equation

²For this reason, the Helmholtz equation is called the time independent wave equation.

³See Jackson section 1.8, but note the change in sign. We're taking the unit normal to S to point *into* the volume V , not outward!



$$\iiint_V (f_o \nabla^2 g_o - g_o \nabla^2 f_o) dV = \oint_S \left(f_o \frac{\partial g_o}{\partial n} - g_o \frac{\partial f_o}{\partial n} \right) dS \quad (1)$$

where $\partial/\partial n$ denotes differentiation inward along a normal to S . Since g is also a solution to the wave equation, g_o must satisfy the Helmholtz equation:

$$(\nabla^2 + k^2) g_o = 0 \quad \text{where} \quad k = \frac{\omega}{c}$$

Therefore $f_o \nabla^2 g_o - g_o \nabla^2 f_o = f_o k^2 g_o - g_o k^2 f_o = 0$ and so the LHS of eqn(1) is zero. Thus, eqn(1) reduces to:

$$\oint_S \left(f_o \frac{\partial g_o}{\partial n} - g_o \frac{\partial f_o}{\partial n} \right) dS = 0 \quad (2)$$

Now take $g_0(x, y, z) = \frac{e^{iks}}{s}$ where s is the distance to P . Because of the singularity at $s = 0$, P must be excluded from the region of integration in the RHS of eqn(1).⁴ Therefore, construct a spherical surface S' of radius ϵ around P . The Green Theorem region of integration will be the region between S and S' , since this excludes the singularity. Applying Green's Theorem to the volume between S and S' :

⁴Even though the LHS is zero, Green's Theorem only applies to functions with continuous derivatives.

$$\oint_S \left(f_o \frac{\partial g_o}{\partial n} - g_o \frac{\partial f_o}{\partial n} \right) dS - \oint_{S'} \left(f_o \frac{\partial g_o}{\partial n'} - g_o \frac{\partial f_o}{\partial n'} \right) dS' = 0 \quad (3)$$

Since S' is spherical, it's unit outward normal $\hat{n}'$ is parallel to $\vec{s}$, and therefore $\partial_{\hat{n}'} g_o$ has a simple form:

$$\frac{\partial g_o}{\partial n'} = \frac{\partial}{\partial n'} \left(\frac{e^{iks}}{s} \right) = \frac{e^{iks}}{s} \left(ik - \frac{1}{s} \right)$$

and so, from eqn(3) we get:

$$\oint_S \left[f_o \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial f_o}{\partial n} \right] dS = - \oint_{S'} \left[f_o \frac{e^{iks}}{s} \left(ik - \frac{1}{s} \right) - \frac{e^{iks}}{s} \frac{\partial f_o}{\partial n'} \right] dS' \quad (4)$$

Because S' is a sphere, $dS' = \epsilon^2 d\Omega$, and eqn(4) becomes:

$$\oint_S \left[f_o \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial f_o}{\partial n} \right] dS = - \oint_{\Omega'} \frac{e^{ik\epsilon}}{\epsilon} \left[f_o \left(ik - \frac{1}{\epsilon} \right) - \frac{\partial f_o}{\partial n'} \right] \epsilon^2 d\Omega' \quad (5)$$

Clearly, the LHS is independent of ϵ . Therefore, but the RHS must also be independent of ϵ too. Because the whole thing is independent of ϵ , let's take the limit of the RHS as $\epsilon \rightarrow 0$.

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} - \oint_{\Omega'} \frac{e^{ik\epsilon}}{\epsilon} \left[f_o \left(ik - \frac{1}{\epsilon} \right) - \frac{\partial f_o}{\partial n'} \right] \epsilon^2 d\Omega' \\ = \lim_{\epsilon \rightarrow 0} - \oint_{\Omega'} e^{ik\epsilon} \left[f_o (ik\epsilon - 1) - \epsilon \frac{\partial f_o}{\partial n'} \right] d\Omega' \\ = \oint_{\Omega'} f_o d\Omega' \\ = 4\pi f_o(P) \end{aligned}$$

That last line comes from the mean value theorem for potentials. See Jackson, problem 1.10, which is a proof of the mean value theorem for potentials. Plugging this result back into eqn(5), we get the Kirchhoff theorem for monochromatic light:

$$f_o(P) = \frac{1}{4\pi} \oint_S \left[f_o \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial f_o}{\partial n} \right] dS \quad (6)$$

This theorem expresses a component of $\vec{E}$ or $\vec{H}$ at point P as a surface integral of the field component over any surface surrounding P . One might wonder about this: we need to know the field before we can calculate the field.

In practise, this is not a problem because we often know the field components on bounding surfaces because a physical situation gives us boundary conditions. Therefore, if we know the boundary conditions on field components, this theorem gives us a method of calculating the field at arbitrary points within the boundary.

1.2 Non-Monochromatic Light

Let $f(x, y, z, t)$ be a multichromatic solution to the electromagnetic wave equation (a component of $\vec{E}$ or $\vec{H}$ with many angular frequencies):

$$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

Since f is multichromatic and we're assuming that it satisfies any continuity conditions that a physical wave must satisfy, it's expressible as a Fourier integral over all angular frequencies:

$$f(x, y, z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_{\omega}(x, y, z) e^{-i\omega t} d\omega \quad (7)$$

Then, by the Fourier inversion formula:

$$F_{\omega}(x, y, z, w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, y, z, t) e^{i\omega t} dt$$

By plugging eqn(7) into the wave equation, we can show that if f satisfies the wave equation, F_{ω} must satisfy the Helmholtz equation. Thus we can apply the Helmholtz theorem to individual Fourier components of F_{ω} :

$$F_{\omega}(P) = F_{\omega}(x, y, z) = \frac{1}{4\pi} \oint_S \left[F_{\omega} \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial F_{\omega}}{\partial n} \right] dS \quad (8)$$

Plugging the Helmholtz expression of F_ω eqn(8) into the Fourier expansion for f eqn(7):

$$\begin{aligned}
f(P, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_\omega(x, y, z) e^{-i\omega t} d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{4\pi} \oint_S \left[F_\omega \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial F_\omega}{\partial n} \right] \right) e^{-i\omega t} dS d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{4\pi} \oint_S \left[F_\omega \frac{\partial}{\partial n} \left(\frac{e^{-i\omega(t-s/c)}}{s} \right) - \frac{e^{-i\omega(t-s/c)}}{s} \frac{\partial F_\omega}{\partial n} \right] dS d\omega \\
&= \frac{1}{4\pi} \oint_S \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F_\omega \frac{\partial}{\partial n} \left(\frac{e^{-i\omega(t-s/c)}}{s} \right) - \frac{e^{-i\omega(t-s/c)}}{s} \frac{\partial F_\omega}{\partial n} \right] d\omega dS \\
&= \frac{1}{4\pi} \oint_S \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F_\omega \left(\frac{\partial}{\partial n} \frac{1}{s} + \frac{i\omega}{sc} \frac{\partial s}{\partial n} \right) \frac{e^{-i\omega(t-s/c)}}{s} - \frac{e^{-i\omega(t-s/c)}}{s} \frac{\partial F_\omega}{\partial n} \right] d\omega dS \\
&= \frac{1}{4\pi} \oint_S \frac{1}{\sqrt{2\pi}} \left(\frac{\partial}{\partial n} \frac{1}{s} \right) \int_{-\infty}^{\infty} F_\omega \frac{e^{-i\omega(t-s/c)}}{s} d\omega - \frac{1}{sc} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} F_\omega \frac{e^{-i\omega(t-s/c)}}{s} d\omega \\
&\quad - \frac{1}{s} \frac{\partial}{\partial n} \int_{-\infty}^{\infty} F_\omega \frac{e^{-i\omega(t-s/c)}}{s} d\omega dS
\end{aligned}$$

Where $\int F_\omega e^{-i\omega(t-s/c)} d\omega$ is the retarded value⁵ of wave equation solution: these are functions that are evaluated at the retarded time $t - s/c$. Therefore, we can write down the general form of Kirchhoff's Theorem:

$$f(P, t) = \frac{1}{4\pi} \oint_S \left[f_{\text{ret}} \frac{\partial}{\partial n} \left(\frac{1}{s} \right) - \frac{1}{sc} \frac{\partial s}{\partial n} \frac{\partial f_{\text{ret}}}{\partial t} - \frac{1}{s} \frac{\partial f_{\text{ret}}}{\partial n} \right] dS \quad (9)$$

Given a solution $f(x, y, z, t)$ (a component of $\vec{E}$ or $\vec{H}$) of the EM wave equation on a closed surface, Kirchhoff's Theorem (eqn(6) for monochromatic and eqn(9) for multichromatic light) gives us f at any point within that surface. For multichromatic light, f within S is expressed with a retarded time variable. Strangely, for monochromatic light f at P is not dependent on retarded time. That doesn't make sense to me.

⁵See any book on electricity and magnetism for an explanation of retarded time.