Problem 3.14

Prove the result (26), pg. 79:

$$\mu_r = \mu'_r - \binom{r}{1}\mu'_{r-1}\mu + \cdots + (-1)^j\binom{r}{j}\mu'_{r-j}\mu_j + \cdots + (-1)^r\mu'_0\mu^r$$

The whole thing starts with expanding $\mu_r = \langle (X-\mu)^r \rangle$:

$$\begin{aligned}
\mu_r &= \langle (X-\mu)^r \rangle \\
&= \left\langle \sum_{j=0}^{r} (-1)^j \binom{r}{j} X^{r-j}\mu^j \right\rangle \\
&= \binom{r}{0}\langle X^r\rangle - \binom{r}{1}\langle X^{r-1}\rangle\mu + \cdots + (-1)^j\binom{r}{j}\langle X^{r-j}\rangle\mu^j \\
&\qquad + \cdots + (-1)^{r-1}\binom{r}{r-1}\langle X^1\rangle\mu^{r-1} + (-1)^r\binom{r}{r}\langle X^0\rangle\mu^r \\
&= \langle X^r\rangle - r\langle X^{r-1}\rangle\mu + \cdots + (-1)^j\binom{r}{j}\langle X^{r-j}\rangle\mu^j \\
&\qquad + \cdots + (-1)^{r-1} r\langle X\rangle\mu^{r-1} + (-1)^r\langle X^0\rangle\mu^r \\
&= \mu'_r - r\mu'_{r-1}\mu + \cdots + (-1)^j\binom{r}{j}\mu'_{r-j}\mu^j \\
&\qquad + \cdots + (-1)^{r-1} r\mu'_1\mu^{r-1} + (-1)^r\mu^r
\end{aligned}$$

← The same expression since $\mu'_0 = \langle X^0\rangle = 1$

The last two terms can be combined by noting $\mu'_1 = \langle X\rangle = \mu$:

$$(-1)^{r-1} r\mu^r + (-1)^r\mu^r = (-1)^{r-1}(r-1)\mu_r$$

to give:

$$\begin{aligned}
\mu_r &= \mu'_r - r\mu'_{r-1}\mu + \cdots + (-1)^j\binom{r}{j}\mu'_{r-j}\mu^j \\
&\qquad + \cdots + (-1)^{r-1}(r-1)\mu_r
\end{aligned}$$