Problem 3.11

The quantity $\langle (X-\alpha)^2 \rangle$ is a minimum when $\alpha = \mu = \langle X \rangle$.
Prove this theorem.

We can find the extrema of $\langle (X-\alpha)^2 \rangle$ by finding the roots of its derivative.

$$\frac{d}{d\alpha} \langle (X-\alpha)^2 \rangle = \left\langle \frac{d}{d\alpha} (X-\alpha)^2 \right\rangle = -2 \langle X-\alpha \rangle$$

$$= -2 \left(\langle X \rangle - \alpha \right)$$

$$= -2 (\mu_x - \alpha) \overset{set}{\equiv} 0$$

$$\Longrightarrow \alpha = \mu_x$$

So $\langle (X-\alpha)^2 \rangle$ has extrema at $\alpha = \mu_x = \langle X \rangle$, but is this extremum a maximum or minimum? For that, we need the 2nd derivative test.

$$\frac{d^2}{d\alpha^2} \langle (X-\alpha)^2 \rangle = -2 \frac{d}{d\alpha} (\mu_x - \alpha) = +2$$

Since the 2nd derivative is always positive, $\langle (X-\alpha)^2 \rangle$ is concave up, therefore, $\alpha = \mu_x = \langle X \rangle$ is a <u>minimum</u> of the function $\langle (X-\alpha)^2 \rangle$.