Notes and Examples of Matrix Topics

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1 No degeneracy, no zero eigenvalues

Consider the Hermitian matrix:

$$\hat{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

1.1 Trace and Determinant

$$\text{Det}[\hat{M}] = -2 \qquad \text{Tr}[\hat{M}] = 2$$

1.2 Eigenvalues and Eigenvectors

The eigenvalue equation is $\text{Det}[\hat{M} - \lambda\hat{I}] = -\lambda^3 + 2\lambda^2 + \lambda - 2$, which yield the eigenvalues $-1, 1, 2$. Based on these eigenvalues, we'd expect that M diagonalized is:

$$\hat{M}_D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The eigenvector equation is:

$$\hat{M}\,|u\rangle = \lambda\,|u\rangle \qquad \Longrightarrow \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b + \sqrt{2}c \\ \sqrt{2}b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

There is nothing surprising or tricky about finding the eigenvectors, so we'll just state them:

$$|\lambda_{-1}\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ -\sqrt{2} \end{pmatrix} \qquad |\lambda_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\lambda_2\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

It's easily seen that the eigenvalues are non-degenerate and the eigenvectors are orthogonal.

1.3 Spectral Decomposition of $\hat{M}$

Let's verify the spectral decomposition of M:

$$\begin{aligned}
\hat{M} &= \sum_{i=0}^{3} \lambda_i \left|\lambda_i\right\rangle \left\langle\lambda_i\right| \\
&= \lambda_1 \left|\lambda_1\right\rangle \left\langle\lambda_1\right| \;+\; \lambda_2 \left|\lambda_2\right\rangle \left\langle\lambda_2\right| \;+\; \lambda_3 \left|\lambda_3\right\rangle \left\langle\lambda_3\right| \\
&= -\frac{1}{3}\begin{pmatrix} 0 \\ 1 \\ -\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 1 & -\sqrt{2} \end{pmatrix} \;+\; 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \;+\; -\frac{2}{3}\begin{pmatrix} 0 \\ \sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} & 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}
\end{aligned}$$

1.4 Diagonalization of $\hat{M}$

Now we'll actually diagonalize eq(1) rather than just writing it down using our knowledge of the eigenvalues. This amounts to expressing eq(1) in the eigenbasis formed by its eigenvalues. Form the transformation matrix with the eigenvectors of M as column vectors:

$$\hat{U}_D = \begin{pmatrix} \left|\lambda_{-1}\right\rangle & \left|\lambda_1\right\rangle & \left|\lambda_2\right\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & \sqrt{3} & 0 \\ 1 & 0 & \sqrt{2} \\ -\sqrt{2} & 0 & 1 \end{pmatrix}$$

2 No Degeneracy, With Zero Eigenvalue

Consider the matrix:

$$\hat{M} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \tag{1}$$

2.1 Trace and Determinant of $\hat{M}$

It's easy seen that:

$$\text{Det}[\hat{M}] \;=\; 0 \qquad\qquad \text{Tr}[\hat{M}] \;=\; 0$$

2.2 Eigenvalues and Eigenvectors of $\hat{M}$

The eigenvalue equation is $\text{Det}[\hat{M} - \lambda I] = \lambda(1 - \lambda^2) = 0$ which yields eigenvalues of $1, 0, -1$. Based on these eigenvalues, we'd expect $\hat{M}$ diagonalized is (we'll verify this shortly):

$$\hat{M}_D \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \tag{2}$$

The eigenvector equation is:

$$\hat{M}\,|u\rangle = \lambda\,|u\rangle \qquad \Longrightarrow \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}b\\ a+c\\ b\end{pmatrix} \;=\; \lambda\begin{pmatrix}a\\ b\\ c\end{pmatrix}$$

Getting the eigenvectors for λ_1 and λ_{-1} is straightforward:

$$\lambda_1 : \quad \frac{b}{\sqrt{2}} = a = c \quad \text{and} \quad \frac{a+c}{\sqrt{2}} = b \qquad \Longrightarrow \qquad |\lambda_1\rangle = \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$$

$$\lambda_{-1} : \quad \frac{b}{\sqrt{2}} = -a = -c \quad \text{and} \quad \frac{a+c}{\sqrt{2}} = -b \qquad \Longrightarrow \qquad |\lambda_{-1}\rangle = \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

The zero eigenvalue is a problem, since it gives an eigenvector with components of zero. What we need to do is find a vector that's orthogonal to both $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$. This is guaranteed to be an eigenvalue since we're working a 3 dimensional vector space:

$$
\begin{aligned}
\langle\lambda_0|\lambda_1\rangle &= \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} &= a + \sqrt{2}b + c = 0 \\
\langle\lambda_0|\lambda_{-1}\rangle &= \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} &= a + -\sqrt{2}b + c = 0
\end{aligned}
$$

Adding these two equations gives the eigenvector of eigenvalue 0:

$$
|\lambda_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
$$

So now we have all the eigenvalues and eigenvectors of eq(1):

$$
|\lambda_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad |\lambda_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad |\lambda_{-1}\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}
$$

2.3 Spectral Decomposition of $\hat{M}$

Let's verify the spectral decomposition of $\hat{M}$:

$$
\begin{aligned}
\hat{M} &= \sum_{i=0}^{3} \lambda_i |\lambda_i\rangle\langle\lambda_i| \\
&= \lambda_1 |\lambda_1\rangle\langle\lambda_1| \;+\; \lambda_2 |\lambda_2\rangle\langle\lambda_2| \;+\; \lambda_3 |\lambda_3\rangle\langle\lambda_3| \\
&= \frac{1}{4}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix} \;+\; 0 \;+\; -\frac{1}{4}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{2} & 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2 & \sqrt{2} \\ 0 & 2\sqrt{2} & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 2 & -\sqrt{2} \\ 0 & -2\sqrt{2} & 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

2.4 Diagonalization of $\hat{M}$

Now we'll actually diagonalize eq(1) rather than just writing it down using our knowledge of the eigenvalues. This amounts to expressing eq(1) in the eigenbasis formed by its eigenvalues. Form the transformation matrix with the eigenvectors of $\hat{M}$ as column vectors:

$$\hat{M}_D = \begin{pmatrix} |\lambda_1\rangle & |\lambda_0\rangle & |\lambda_{-1}\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \tag{3}$$

And calculating the similarity transformation based on eq(3) confirms the diagonalized version of $\hat{M}$ that we wrote down earlier:

$$\begin{aligned} \hat{M}_D &= U_D^\dagger M U_D \\ &= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

verifyng the eigenvalues we found as well as $\hat{M}_D$ that we wrote down eq(2) based on them. In addition, it's easily seen that the determinant and trace of $\hat{M}_D$ are both 0, as they were for $\hat{M}$.

2.5 Trace and Determinant of $\hat{M}_D$

It's easily seen that:

$$\text{Det}[\hat{M}]_D = 0 \qquad \text{Tr}[\hat{M}] = 0$$

As expected, the trace and determinant of $\hat{M}$ and $\hat{M}_D$ are the same.

2.6 Eigenvalues and Eigenvectors of $\hat{M}_D$

The eigenvalue equation for $\hat{M}_D$ is $\text{Det}[\hat{M}_D - \lambda I] = (1-\lambda)\lambda(1+\lambda^2) = 0$ which yields eigenvalues of $1, 0, -1$. As expected, the eigenvalues of $\hat{M}$ and $\hat{M}_D$ are the same.

The eigenvector equation is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The first and last eigenvectors are easily seen to be:

$$|\lambda_1\rangle : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |\lambda_{-1}\rangle : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The 2nd eigenvector is trivial too. We're working in a 3 dimensional vector space, so *any* vector which is orthogonal to both $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ will work. There is an obvious choice here:

$$|\lambda_1\rangle : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\lambda_0\rangle : \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\lambda_{-1}\rangle : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2.7 Spectral Decomposition of $\hat{M}_D$

This one is so easy that you can picture it in your head!

$$\begin{aligned} \hat{M}_D &= \sum_{i=0}^{3} \lambda_i \, |\lambda_i\rangle \langle\lambda_i| \\ &= \lambda_1 \, |\lambda_1\rangle \langle\lambda_1| \; + \; \lambda_2 \, |\lambda_2\rangle \langle\lambda_2| \; + \; \lambda_3 \, |\lambda_3\rangle \langle\lambda_3| \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \; + \; 0 \; - \; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

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