Dirac Notation for Smart Engineers

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1 Vectors as Coordinate Independent Objects

Think of a California map with a Cartesian coordinate system centered on Los Angeles with x pointing east and y pointing north. Draw a vector $\vec{r}$ from LA to Sacramento. This arrow shows the location of Sacramento with respect to LA. But how would describe the location of Sacramento if you don't have the map in front of you?

You could give the x and y Cartesian coordinates of Sacramento, like: "walk $69\,mi$ east and $394\,mi$ north".

You could also give the r and θ polar coordinates, like: "walk $400\,mi$ at 80^o north of east". Both (x, y) and (r, θ) describe the same vector, so is $(69\,mi,\ 394\,mi)$ really the vector? Or would you call the vector $(400\,mi,\ 80^o)$?

Neither. Neither set of numbers *is* the vector. The vector has a reality all to its own, and the two sets of numbers are both simply *representations* of the vector. And before you start to say I'm splitting hairs, let me say that this turns out to be very important when you want to talk generally about a specific physical system. You can talk very abstractly about $\vec{r}$ without throwing it into a coordinate system by talking about (x, y) and (r, θ).

The California map is a two dimensional vector space. Any position can be described by two linearly independent basis vectors, usually normalized, but they don't have to be.

The Cartesian unit vectors $\hat{x}$ and $\hat{y}$ are the usual choice for a 2-D vector space. The position vector between LA and Sacramento can be written in Cartesian coordinates as:

$$\vec{r} = x\hat{x} + y\hat{y}$$

We can find the x and y components by using the dot product:

$$x = \vec{r} \cdot \hat{x} = r_x \qquad \text{and} \qquad y = \vec{r} \cdot \hat{y} = r_y$$

2 Enter Dirac Notation

In Dirac notation, a vector, like $\vec{r}$, is called a 'ket' and is denoted by $|r\rangle$. So as a first step in converting to Dirac notation, we can write:

$$|r\rangle \;=\; r_x\,|x\rangle + r_y\,|y\rangle \;=\; 69\,mi\,|x\rangle + 394\,mi\,|y\rangle$$

where $|x\rangle = \hat{x}$ and $|y\rangle = \hat{y}$ are basis vectors for the map of California. But there's more.

Every vector space V has an associated vector space V^*, called the *dual space* of V. Vectors live in V, but *dual vectors* live in V^*. There's a 1-1 and onto mapping between vectors in V and dual vectors in V^*. Dual vectors are called *bras* and denoted by $\langle v|$.

There's a unique correspondance between kets in V and bras in V^*, so we name them similarly: The ket $|r\rangle$ has the unique bra $\langle r|$. The bra $\langle x|$ has a unique ket $|x\rangle$, both of which can have many representations. You can think of kets as column vectors:

$$|r\rangle \leftrightarrow \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} 69\,mi \\ 394\,mi \\ 0\,mi \end{pmatrix}$$

The symbol $\leftrightarrow$ means "is represented by": $|r\rangle$ is not the column 3-tuple (r_x, r_y, r_z), but in a particular basis, it can be represented as such. Similarly, bras can be thought of as row vectors. To form a bra from a ket, you turn the column vector into a row vector, and take the complex conjugate of each element. Since $|r\rangle$ has real elements, we can write:

$$\langle r| \leftrightarrow \begin{pmatrix} r_x, & r_y, & r_z \end{pmatrix} = \begin{pmatrix} 69\,mi, & 394\,mi, & 0\,mi \end{pmatrix}$$

You can use Dirac notation wherever you use vector notation. It's often more convenient than standard vector notation, but sometimes not. Consider associativity. Personally, I think $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ is prettier than $(|a\rangle + |b\rangle) + |c\rangle = |a\rangle + (|b\rangle + |c\rangle)$.

3 Vectors

It's an unfortunate thing that the word "vector" has two common uses. Most of us learn that a vector is an n-tuple, like $\vec{v} = 3\hat{x} - 2\hat{y}$. However, that's not quite correct: an n-tuple is an example of a vector, but a vector is actually a more general thing.

If you've studied linear algebra, you know about vector spaces. A vector is simply an element of a vector space. For a vector space of matrices, a vector is a matrix. For vector spaces of functions, vectors are functions. For vector spaces of n-tupes, vectors are n-tuples.

4 Inner Products

There's a general operation called an *inner product* which takes two vectors (a general vector) and assigns to them a complex or real number. The inner product is general and there's a lot of leeway in how you can define it, there are certain rules that inner products need to follow, like linearity, meaning $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

For instance, if V is the square integrable functions, we can define an inner product as:

$$f(x) \cdot g(x) = \int_{-\infty}^{\infty} f(x)^* g(x)\ dx$$

If our vector space is the space 3-tuples, we could define the inner product as:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

which is the familiar dot product: so the dot product is an example of an inner product. We don't *have* to define an inner product of n-tuples this way. We *could* define it this way:

$$\vec{a} \cdot \vec{b} = -a_x b_x + 2a_y b_y + \pi a_z b_z$$

I'll be using the term inner product, but if you're uncomfortable with that, you can mentally translate it into the dot product, which you may be more comfortable with.

In Dirac notation, an inner product takes bras and kets into $\mathbb{R}$ (or $\mathbb{C}$) and is written as:

$$\langle a|b\rangle = c \in R$$

Colloquially, you take a ket, right multiply it with a bra, and you get a real number called the inner product of the bra and ket. So instead of writing $\hat{x} \cdot \vec{r} = r_x$, we now write:

$$\langle x|r\rangle = r_x$$

Similarly, an inner product between $\hat{y}$ and $\vec{r}$ is denoted by $\langle y|r\rangle = r_y$.

Just so that there's no confusion, the Dirac expression of commutativity ($\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$) looks like $\langle a|b\rangle = \langle b|a\rangle$. Don't confuse this with $\langle a|b\rangle \neq |b\rangle\langle a|$. This is wrong.[1]

Let's talk about quantum mechanics, since that's where I wanted to go with all of this.

[1] Actually, $|b\rangle\langle a|$ has a meaning, which I may or may not get to in this paper.

5 State Vectors

A state vector, often denoted as $|\Psi\rangle$, is a vector of data which gives all the information there is to know about something within a particular context. For instance, to air traffic controllers, $|\Psi\rangle$ could be a vector containing the coordinates of all the planes in the airspace covered by his radar. As planes move around, $|\Psi\rangle$ continuously changes. Note that $|\Psi\rangle$ can be expressed in Cartesian, Polar, Cylindrical, Hyperbolic Spheroid, etc. coordinates. But no matter how you represent it, it's still the same vector.

There are discrete state vectors too, like the state vector of World Chess Federation champion rankings, which could only change after a chess match. To stretch the analogy to the limits, you can express it in different "bases" too: an ordered list of names (the name basis), social security numbers (the SSN basis) or driving licenses (the license basis). Anyway.

In QM, $|\Psi\rangle$ is a wavefunction describing all there is to know about a system like a hydrogen atom or a particle in a potential well. The quantum mechanical state vector (hereafter just 'state vector' since I'm done with my silly examples) of a system is a vector that lives in an infinite dimensional vector space called a Hilbert space. Hilbert space has an infinite number of basis vectors, and there's an infinite number of ways to choose them.

Dirac notation is especially useful here because it provides a way for us to refer to the state vector without having to explicitly pick a basis.

Ignoring spin, Hilbert space is the collection of all square integrable functions[2]. We'll talk about two popular choices for a basis of Hilbert space and express $|\Psi\rangle$ in terms of them.

One basis for Hilbert space is the set of all delta functions, $\delta(x - x_0)$, called the position basis. In Dirac notation these basis vectors are written as $|x_0\rangle$. As discussed, $\langle x_0|\Psi\rangle$ is the component of $|\Psi\rangle$ along the direction defined by $|x_0\rangle$. Since $|\Psi\rangle$ can be represented by all these components, so we can make an identification between the state vector and the set of its components in position space. That is, the state vector $|\Psi\rangle$ expressed in the position basis is $\Psi(x)$ and each of its infinite components can be found by calculating $\langle x|\Psi\rangle$.

Another basis for Hilbert space is the set of complex exponentials (called momentum basis):

$$|p\rangle = \frac{1}{\sqrt{2\pi}\hbar} e^{-ipx/\hbar}$$

Components of $|\Psi\rangle$ in this basis are obtained by taking the dot product $\langle p|\Psi\rangle$ which is the component of $|\Psi\rangle$ in the direction of $|p\rangle$. In this basis, the state vector is denoted by $\Psi(p)$.

[2]The collection of all functions $f(x)$ which have the property that $\int_{-\infty}^{\infty} |f(x)|^2 \, dx$ is finite.

6 Exercise

Let $f(x)$ be a square integrable function. We can expand such a function in terms of a set of 'orthogonal functions' which we'll collectively call $\{\alpha_i\}$. If the $\{\alpha_i\}$ are powers of x, the expansion is called a Taylor series. If they're sines and cosines, it's called a Fourier series. If they're Bessel functions, it's called arcane. The following notation indicates $f(x)$ is expanded by some unspecified set of orthogonal functions $\{\alpha_i\}$:

$$f(x) = |\Psi\rangle = \alpha_0 |\alpha_0\rangle + \alpha_1 |\alpha_1\rangle + \alpha_2 |\alpha_2\rangle + \dots$$

Where $|\alpha_j\rangle$ is the j'th orthogonal function and α_j is the coefficient of the j'th term of the expansion. Since the $\{\alpha_i\}$ are orthogonal, they have the property that:

$$\int \alpha_i^\dagger \alpha_j \; dx = \langle \alpha_i | \alpha_j \rangle = w \delta_{ij}$$

If the $\{\alpha_i\}$ are *normalized* as well as orthogonal, then w is unity. Just to make things a bit more fun, assume w isn't unity. Work in Dirac notation unless I tell you otherwise.

1. Use orthogonality to find α_5.
2. Give a general formula for α_i in bothdirac notation *and* an integral involving $f(x)$.
3. Let $f(x) = \sin(x)$ and suppose we've expanded it in terms of a Fourier series; eg. $\{\alpha_n\} = \{\sin(nx)\}$. Use your formula to calculate the first 3 α_i.
4. Repeat the last question, but now assume that $f(x)$ is expanded in a Taylor Taylor series; eg. $\{\alpha_n\} = \{x^n\}$.

If you have questions or comments, please direct them to me, P. J. Salzman at `p@dirac.org`. If I'm not too busy, I'll try to answer them (but no promises).